MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 4

DUE FRIDAY, MAY 3

Problem 1. For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(1) $V = \mathbb{R}^2$, $T(a, b) = (2a + b, a - 3b)$ and $x = (3, 5)$.
(2) $V = \mathbb{C}^2$, $T(z_1, z_2) = (2z_1 + iz_2, (1 - i) z_1)$ and $x = (3 - i, 1 + 2i)$.
(3) $V = P_1(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$, $T(f) = f' + 3f$, $f(t) = 4 - 2t$.

Problem 2. Let T be a linear operator on an inner product space V.

(1) Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.
(2) Prove that $T^*T = T_0$ implies $T = T_0$.
Is the same result true if we assume that $TT^* = T_0$?

Problem 3. Give an example of a linear operator T on an inner product space V such that $N(T) \neq N(T^*)$.

Problem 4. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = \left(T^{-1}\right)^*$.

Problem 5. Let V be a finite-dimensional vector space.

Definition. If $V = W_1 \oplus W_2$, then a linear operator T on V is the *projection on* W_1 *along* W_2 if, whenever $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, then we have $T(x) = x_1$.

(1) Show that $R(T) = W_1$ and $N(T) = W_2$.
Hence $V = R(T) \oplus N(T)$. We will say that T *is a projection* if this equality holds (in which case it is a projection on $R(T)$ along $N(T)$).
(2) Prove that $T \in \mathcal{L}(V)$ is a projection if and only if $T = T^2$.

Problem 6. Let V be a finite dimensional inner product space, and let W be a subspace.

(1) Prove that $V = W \oplus W^{\perp}$.
(2) Show that if T is a projection on W along $W^{\perp}$, then $T = T^*$.

Problem 7. Let T be a linear operator on an inner product space V. Prove that $\|T(x)\| = \|x\|$ for all $x \in V$ if and only if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$.

Problem 8. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove the following results.

(1) $R(T^*)^\perp = N(T)$ and $R(T^*) = N(T)^\perp$.
(2) $N(T^*T) = N(T)$, and deduce from it that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.
(3) $\operatorname{rank}(T) = \operatorname{rank}(T^*)$, and deduce from (2) that $\operatorname{rank}(TT^*) = \operatorname{rank}(T)$.
(4) For any $n \times n$ matrix A, $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$.