MATH 115B (CHERNIKOV), SPRING 2019

PROBLEM SET 2

DUE FRIDAY, APRIL 19

Problem 1. For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

(1) $V = P_3(\mathbb{R}), T(f(x)) = f'(x), W = P_2(\mathbb{R})$.
(2) $V = P(\mathbb{R}), T(f(x)) = xf(x), W = P_2(\mathbb{R})$.
(3) $V = \mathbb{R}^3, T(a,b,c) = (a+b+c, a+b+c, a+b+c)$, and $W = \{(t,t,t) : t \in \mathbb{R}\}$.
(4) $V = C([0,1]), T(f(t)) = \left[\int_0^1 f(x)\,dx\right] \cdot t, W = \{f \in V : f(t) = at + b \text{ for some } a, b \in \mathbb{R}\}$.
(5) $V = M_{2\times 2}(\mathbb{R}), T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A, W = \{A \in V : A^t = A\}$.

Problem 2. For each linear operator T on the vector space V find an ordered basis for the T-cyclic subspace generated by the vector z.

(1) $V = \mathbb{R}^4, T(a,b,c,d) = (a+b, b-c, a+c, a+d), z = e_1$.
(2) $V = P_3(\mathbb{R}), T(f(x)) = f''(x), z = x^2$.
(3) $V = M_{2\times 2}(\mathbb{R}), T(A) = A^t, z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
(4) $V = M_{2\times 2}(\mathbb{R}), T(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A, z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Problem 3. For each linear operator T and cyclic subspace W in Problem 2 compute the characteristic polynomial of T_W.

Problem 4. Let V and W be non-zero finite dimensional vector spaces over the same field F, and let $T : V \to W$ be a linear transformation.

(1) Prove that T is onto if and only if T^t is one-to-one.
(2) Prove that T^t is onto if and only if T is one-to-one.

Problem 5. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -a_{k-2} \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix},$$

with $a_0, \dots, a_{k-1}$ arbitrary scalars in F. Prove that the characteristic polynomial of A is

$$(-1)^k \left(a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k\right).$$

(Hint: use induction on k, expanding the determinant along the first row.)

Problem 6. Let T be a linear operator on a finite-dimensional vector space V.

(1) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
(2) Deduce that if the characteristic polynomial of T splits, then any non-trivial T-invariant subspace of V contains an eigenvector of T.

Problem 7.

(1) Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 \ldots + v_k$ is in W, then $v_i \in W$ for all i.
(Hint: use induction on k.)
(2) Suppose that $\dim(V) = n$ and T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself.
(Hint: use (1) to find a vector v such that $\left\{v, T(v), \ldots, T^{n-1}(v)\right\}$ is linearly independent.)

Problem 8. Prove that the restriction of a diagonalizable linear operator T to any non-trivial T-invariant subspace is also diagonalizable.
(Hint: use Problem 7(1).)