MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 9

DUE THURSDAY, JUNE 08

Problem 1. Do Exercise 1, Section 6.1. Justify each answer.

Problem 2. Show that each of the following is **not** an inner product on the given vector spaces.

(1) $\langle (a,b), (c,d) \rangle = ac - bd$ on $\mathbb{R}^2$.
(2) $\langle A, B \rangle = \operatorname{tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$.
(3) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)\, g(t)\, dt$ on $P(\mathbb{R})$, where $'$ denotes differentiation.

Problem 3. Determine if there is an inner product on $\mathbb{R}^2$ such that the associated norm satisfies $\|(x,y)\| = |x| + |y|$ for all $x, y \in \mathbb{R}$. In either of the cases, justify it with a proof.

Problem 4. Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$ from it.

Problem 5. Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is injective.

Problem 6. Do Exercise 1, Section 6.2. Justify each answer.

Problem 7. Apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $\operatorname{Span}(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $\operatorname{Span}(S)$.

(1) $V = \mathbb{R}^3$ with the dot product and $S = \{(1,0,1), (0,1,1), (1,3,3)\}$.
(2) $V = P_2(\mathbb{R})$ with the inner product $\langle f(x), g(x) \rangle = \int_0^1 f(t)\, g(t)\, dt$ and $S = \{1, x, x^2\}$.
(3) $V = M_{2\times 2}(\mathbb{R})$ with the Frobenius inner product and

$$S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}.$$

Problem 8. Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in W^{\perp}$ if and only if $\langle z, v \rangle = 0$ for every $v \in \beta$.