MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 4

DUE THURSDAY, MAY 04

Problem 1. Do Exercise 1, Section 2.1. Justify each answer!

Problem 2. Suppose $T : V \to W$ is a linear transformation of vector spaces. Let V' be a subspace of V, and let W' be a subspace of W.

(1) Prove that $T(V')$ is a subspace of W.
(2) Prove that $\{v \in V : T(v) \in W'\}$ is a subspace of V.

Problem 3. Let V and W be finite dimensional vector spaces, and let $T : V \to W$ be a linear transformation.

(1) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.
(2) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

Problem 4. Let V, W be vector spaces, and suppose $T : V \to W$ is a linear transformation. Suppose moreover that T is bijective.

Prove that if $\{v_1, \ldots, v_n\}$ is a basis for V, then $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W.

Problem 5.

(1) Give an example of distinct linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.
(2) Give an example of two distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.

Problem 6. For each of the following linear transformations, compute $[T]_\beta^\gamma$.

(1) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$, for β and γ the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.
(2) $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$, for $\beta = \gamma$ the standard basis in $\mathbb{R}^n$.
(3) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T(A) = A^t$ for $\beta = \gamma$ the standard basis in $M_{2\times 2}(\mathbb{R})$ (i.e. $\beta = \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$).
(4) $T : P_2(\mathbb{R}) \to \mathbb{R}$ defined by $T(f(x)) = f(2)$ for the standard bases $\beta = \{1, x, x^2\}$ and $\gamma = \{1\}$.
(5) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2$, for β the standard basis of $M_{2\times 2}(\mathbb{R})$ and γ the standard basis of $P_2(\mathbb{R})$.

Problem 7. Let V, W be vector spaces, and let T, U be non-zero linear transformations from V to W. Prove that if $R(T) \cap R(U) = \{0\}$, then T and U are linearly independent vectors in $\mathcal{L}(V, W)$.

Problem 8. Prove Theorem 2.10.