Midterm 2 Review Sheet

Linear transformations

Def Let V and W be v.s. over the same field of scalars F.
A lin. transformation from V to W is a function $T: V \to W$ satisfying
1) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$.
2) $T(cx) = cT(x)$ for all $x \in V$ and $c \in F$.

Properties of lin. transformations
1) Let $T: V \to W$ be a lin. transf. Then:
a) $T(0) = 0$
b) $T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$ for all $x_i \in V$, $a_i \in F$.

2) A function $T: V \to W$ is a lin. transf. $\iff$ $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$, $c \in F$.

Thm 2.6 Let V, W be v.s. v.s. over a field F, and let $\{v_1, \dots, v_n\}$ be a basis for V.
Then for any vectors $w_1, \dots, w_n \in W$ there exists exactly one lin. transf. $T: V \to W$ s.t.
$T(v_i) = w_i$ for $1 \le i \le n$.

Def Let $T: V \to W$ be a lin. transf.
1) T is injective if $T(v) = T(u)$ imlies $v = u$, for all $u, v \in V$.
2) T is surjective if for every $w \in W$ there is some $v \in V$ s.t. $T(v) = w$.
3) T is bijective if it is both injective and surjective.

Null space and range

Def Let V, W be v.s. and $T: V \to W$ a lin. transf.
1) The null space of T is defined as
$N(T) = \{x \in V : T(x) = 0\}$.
2) The range of T is the image of V under T, that is the set
$R(T) = \{y \in W : y = T(x) \text{ for some } x \in V\}$.

Thm 2.1
1) $N(T)$ is a subspace of V.
2) $R(T)$ is a subspace of W.

Thm 2.4 Let $T: V \to W$ be a lin. transf.
1) T is injective $\iff$ $N(T) = \{0\}$.
2) T is surjective $\iff$ $R(T) = W$.

Thm 2.2 Let $T: V \to W$ be a lin. transf.
Assume $\beta = \{v_1, \dots, v_n\}$ is a basis for V.
Then $R(T) = \mathrm{Span}(\{T(v_1), \dots, T(v_n)\})$.

Thm 2.3 (Dimension Theorem)
Let V, W be v.s., $T: V \to W$ a lin. transf., and $\dim(V) < \infty$. Then
$\dim(V) = \dim(N(T)) + \dim(R(T))$.

Thm 2.5 Let $T: V \to W$ be a lin. transf., and assume $\dim(V) = \dim(W)$.
Then the following are equivalent:
1) T is injective.
2) T is surjective.
3) T is bijective.
4) $\dim(R(T)) = \dim(V)$.

The vector space of linear transformations $\mathcal{L}(V,W)$.

Def. Let V, W be v.s. over F, and let $T, U: V \to W$ be linear transformations.
Then we define the functions $T+U$ and aT, for every $a \in F$, by:
$(T+U)(x) = T(x) + U(x)$ for all $x \in V$.
$(aT)(x) = a \cdot T(x)$ for all $x \in V$.

Thm 2.7
If T and U are linear, then $T+U$ and aU are also linear.

Def We denote the set of all lin. transf.'s from V to W by $\mathcal{L}(V,W)$.
Then it is a v.s. over F, with the operations of addition and scalar multiplication described above.
When $W = V$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V,V)$.

Matrix representation of a lin. transf.

Def Let V be a v.s. with $\dim(V) < \infty$. An ordered basis for V is a basis for V with a specified order on its vectors.

Def Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for V.
Then any vector $x \in V$ can be written as
$x = a_1 v_1 + \dots + a_n v_n$ for some unique scalars $a_1, \dots, a_n \in F$.
We define the coordinate vector of x relative to β by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

Def Let V, W be v.s. with ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$, respectively.
Let $T: V \to W$ be a lin. transformation.
Then the matrix representation of T in the ordered bases β and γ is defined as the matrix $[T]_\beta^\gamma \in M_{m \times n}(F)$ given by

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{pmatrix},$$

where $[T(v_i)]_\gamma$ are the coordinates of the vector $T(v_i) \in W$ with respect to the basis γ.
If $V = W$ and $\beta = \gamma$, we simply write $[T]_\beta$.

Thm 2.8
Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T, U: V \to W$ be lin. transformations. Then:
1) $U = T$ (meaning that $U(x) = T(x)$ for all $x \in V$) $\iff [U]_\beta^\gamma = [T]_\beta^\gamma$.
2) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.
3) $[aT]_\beta^\gamma = a \cdot [T]_\beta^\gamma$ for all $a \in F$.

Composition of lin. transf's and matrix multiplication

Def Let V, W, Z be v.s.'s over F. Let $T: V \to W$ and $U: W \to Z$ be lin. transf.'s.
Their composition is the function UT, from V to Z, defined by
$(UT)(x) = U(T(x))$ for all $x \in V$.

Thm 2.9 If T and U are linear, then UT is also linear.

Def Given matrices $A \in M_{m\times n}(F)$ and $B \in M_{n\times p}(F)$, we define the product $AB \in M_{m\times p}(F)$ to be the matrix with the entries

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj}, \quad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

Thm 2.11 Let V, W, Z be fin. dim. v.s.'s with ordered bases α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be lin. transformations. Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.

Corollary Let V be a fin. dim. v.s. with an ordered basis β.
Let $T, U \in \mathcal{L}(V)$.
Then $[UT]_\beta = [U]_\beta [T]_\beta$.

Thm 2.12 (Properties of matrix multiplication).
Let $A \in M_{m\times n}(F)$, $B, C \in M_{n\times p}(F)$, and $D, E \in M_{q\times m}(F)$.
1) $A(B+C) = AB + AC$ and $(D+E)A = DA + EA$.
2) $a(AB) = (aA)B = A(aB)$ for any $a \in F$.
3) $I_m A = A = A I_n$,
where $I_k \in M_{k\times k}(F)$ denotes the identity matrix, $I_k = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.
4) $(AB)^t = B^t A^t$
5) If $\dim(V) = n$ and $I_V: V \to V$ is the identity transf. on V, then $[I_V]_\beta = I_n$ for any ordered basis β for V.

However, matrix multiplication is not commutative, that is $AB \ne BA$ in general.

Thm 2.14 Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T: V \to W$ be a lin. transf. Then for each vector $u \in V$ we have:
$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.
(so, we calculate the coordinates of the vector $T(u)$ from the coordinates of the vector u).

Def To every matrix $A \in M_{m\times n}(F)$, we associate a linear transformation $L_A: F^n \to F^m$ defined by
$L_A(x) = Ax$ for every (column) vector $x \in F^n$.
We call L_A the left-multiplication transformation

Thm 2.15 (Properties of L_A)
Let $A \in M_{m\times n}(F)$.
If $B \in M_{m\times n}(F)$ and β, γ are the standard ordered bases for F^n and F^m, resp., then:
a) $[L_A]_\beta^\gamma = A$.
b) $L_A = L_B \iff A = B$.
c) $L_{A+B} = L_A + L_B$, $L_{aA} = a \cdot L_A$ for all $a \in F$.
d) If $T: F^n \to F^m$ is lin., then there is a unique $C \in M_{m\times n}(F)$ s.t. $T = L_C$. In fact, $C = [T]_\beta^\gamma$.
e) If $E \in M_{n\times p}(F)$, then $L_{AE} = L_A L_E$.
f) If $m = n$, then $L_{I_n} = I_{F^n}$.

Thm 2.16 For $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$, $C \in M_{p\times r}(F)$,
$A(BC) = (AB)C$.
(so matrix multiplication is associative).

Invertibility

Def Let V, W be v.s.'s and $T: V \to W$ linear.
1) A lin. transf. $U: W \to V$ is the inverse of T if
$UT = I_V$ and $TU = I_W$.
2) T is invertible if it has an inverse.

Basic facts
1) If T is invertible, then its inverse is <u>unique</u>, and is denoted by T^{-1}.
2) T is invertible $\iff$ T is a bijection.
3) If T, U are invertible, then
- $(TU)^{-1} = U^{-1}T^{-1}$
- $(T^{-1})^{-1} = T$.

Lemma Let $T: V \to W$ be lin. and invertible, and $\dim(V) < \infty$.
Then $\dim(V) = \dim(W)$.

Def A matrix $A \in M_{n\times n}(F)$ is invertible if there exist $B \in M_{n \times n}(F)$ s.t. $AB = BA = I$.
If such a B exists, the it is <u>unique</u>, called the inverse of A and denoted by A^{-1}.

Thm 2.18 Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T: V \to W$ be lin.
Then T is invertible $\iff$ the matrix $[T]_\beta^\gamma$ is invertible.
Furthermore, $[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.

Isomorphisms

Def Two v.s.'s V and W are isomorphic if there exists an invertible lin. transf. $T: V \to W$. Such a T is called an isomorphism from V onto W.

Thm 2.19 Two fin. dim. v.s.'s V and W are isomorphic $\iff$ $\dim(V) = \dim(W)$.

Corollary. Let V be a v.s. over F. Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Thm 2.20 Let V, W be v.s.'s over F, $\dim(V) = n$, $\dim(W) = m$.
Let β, γ be ordered bases for V, W, resp.
Then the map $\phi: \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by
$$\phi(T) = [T]_\beta^\gamma \quad \text{for all } T \in \mathcal{L}(V,W)$$
is an isomorphism.

Corollary If $\dim(V) = n$, $\dim(W) = m$ then $\dim\left(\mathcal{L}(V,W)\right) = \dim\left(M_{m\times n}(F)\right) = mn$.

Change of coordinate matrix

Def Let V be a fin. dim. v.s., and let β and β' be two ordered bases for V.
Then we define the change of coordinates matrix (changing β'-coords. to β-coords) to be $[I_V]_{\beta'}^{\beta}$.

Thm 2.22
1) $[I_V]_{\beta'}^{\beta}$ is invertible, and $\left([I_V]_{\beta'}^{\beta}\right)^{-1} = [I_V]_{\beta}^{\beta'}$.
2) For any vector $v \in V$,
$[v]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'}$. — so we calculate β-coords. of the vector v from its β'-coords.

Def A lin. transf. $T: V \to V$ is called a lin. operator on V.

Thm 2.23 Let T be a lin. operator on a fin. dim. v.s. V.
Let β, β' be ordered bases for V.
Let $Q = [I_V]_{\beta'}^{\beta}$ be the matrix changing β'-coord's to β-coords. Then
$[T]_{\beta'} = Q^{-1} [T]_\beta Q = [I_V]_\beta^{\beta'} [T]_\beta [I_V]_{\beta'}^{\beta}$

Determinants

Def Let $A \in M_{n \times n}(F)$.
1) For any $1 \le i, j \le n$, let $\tilde{A}_{i,j} \in M_{(n-1)\times(n-1)}(F)$ be the matrix obtained from A by deleting row i and column j.
2) The determinant of A, denoted $\det(A)$, is a scalar in F defined recursively as follows.
For $n=1$: $A = (A_{11})$, we define $\det(A) = A_{11}$.
For $n \ge 2$: we define
$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{i,j} \cdot \det(\tilde{A}_{i,j}) \quad \text{for any } 1 \le i \le n$$
or
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\tilde{A}_{ij}) \quad \text{for any } 1 \le j \le n.$$

Properties of the determinant
Let $A \in M_{n\times n}(F)$.
If $B \in M_{n \times n}(F)$ is the matrix obtained from A by
1) switching two rows (or two columns), then
$\det(B) = -\det(A)$.
2) multiplying a row (or a column) by a scalar $c \in F$, then
$\det(B) = c \cdot \det(A)$
3) adding a multiple of row i to row j (or a multiple of a column i to column j), then
$\det(B) = \det(A)$.

These operations are called elementary row operations.

Fact 1 Using elementary row operations, any square matrix can be transformed into an upper triangular matrix. That is, a matrix of the form
$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ 0 & & \ddots & \vdots \\ & & & A_{nn} \end{pmatrix}.$$

Fact 2 If $A \in M_{n\times n}(F)$ is upper triangular, then
$\det(A) = A_{11} \cdot A_{22} \cdot \ldots \cdot A_{nn}$.

Facts 1 and 2 can be used together to simplify calculating the determinants.

More properties of det.
4) If $B \in M_{n \times n}(F)$, then
$\det(AB) = \det(A) \cdot \det(B) = \det(BA)$.
5) A is invertible $\iff \det(A) \neq 0$. Furthermore,
$\det(A^{-1}) = \det(A)^{-1}$.
6) $\det(A) = \det(A^t)$.

Eigenvectors and eigenvalues

Def A matrix $A \in M_{n\times n}(F)$ is diagonal if $A_{ij}=0$ for all $i\neq j$.

Def A lin. operator $T\in\mathcal{L}(V)$, $\dim(V)<\infty$, is diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Def 1) Given $A,B\in M_{n\times n}(F)$, we say that A and B are similar if there exists an invertible matrix $Q\in M_{n\times n}(F)$ such that $B=Q^{-1}AQ$.
2) A matrix $A\in M_{n\times n}(F)$ is diagonalizable if A is similar to a diagonal matrix.

Thm Let $T\in\mathcal{L}(V)$, $\dim(V)<\infty$ and β,γ ordered bases for V.
Then $\det([T]_\beta)=\det([T]_\gamma)$.
Thus, $\det([T]_\beta)$ does not depend on β, and is called the determinant of T, or $\det(T)$.

Proposition
1) T is bijective $\Leftrightarrow$ $\det T\neq 0$.
2) T is bijective $\Rightarrow$ $\det(T^{-1})=(\det T)^{-1}$.
3) If $U\in\mathcal{L}(V)$, then $\det(TU)=\det(T)\cdot\det(U)$.

Thm Let $T\in\mathcal{L}(V)$, $\dim(V)<\infty$ and β an ord. basis for V. Then:
T is diagonalizable $\Leftrightarrow$ $[T]_\beta$ is diagonalizable.

Corollary $A\in M_{n\times n}(F)$ is diagonalizable $\Leftrightarrow$ L_A is diagonalizable.

Def 1) Let $T\in\mathcal{L}(V)$, $\dim(V)<\infty$.
A non-zero vector $v\in V$ is an eigenvector of T if $T(v)=\lambda v$ for some $\lambda\in F$.
Then λ is an eigenvalue of T corresponding to the eigenvector v.
2) Let $A\in M_{n\times n}(F)$.
A non-zero $v\in F^n$ is an eigenvector of A if $Av=\lambda v$ for some $\lambda\in F$.
Then λ is an eigenvalue of A corresp. to v.

Thm A lin. operator $T\in\mathcal{L}(V)$ is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors for T.
Furthermore, then

$$[T]_\beta=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{pmatrix},$$

where λ_i is the eigenvalue corresponding to the i^{th} vector in β.

Thm 5.2 $\lambda\in F$ is an eigenvalue of T $\Leftrightarrow$ $\det(T-\lambda I_V)=0$.

Corollary For $A\in M_{n\times n}(F)$, $\lambda\in F$ is an eigenvalue of A $\Leftrightarrow$ $\det(A-\lambda I_n)=0$.

Def 1) For $A\in M_{n\times n}(F)$, the polynomial $f(t)=\det(A-tI_n)$ is called the characteristic polynomial of A.
2) Let V be a v.s., $\dim(V)=n$ and β is an ord. basis for V. Let $T\in\mathcal{L}(V)$
We define the characteristic polynomial of T to be $f(t)=\det([T]_\beta-tI_n)$

Properties of char. polynomials
Let $A\in M_{n\times n}(F)$ be given, let $f(t)$ be its char. polynomial.
1) $f(t)$ is of degree n, and moreover $f(t)=(-1)^n t^n+c_{n-1}t^{n-1}+\ldots+c_0$ for some $c_0,\ldots,c_{n-1}\in F$.

2) $\lambda \in F$ is an eigenvalue of $A \iff f(\lambda)=0$ (that is, λ is a root of $f(t)$).
3) A has at most n distinct eigenvalues (as $f(t)$ has at most n roots).
4) If $\lambda \in F$ is an eigenvalue of A, then:
$x \in F^n$ is an eigenvector of A corresp. to $\lambda \iff x \neq 0$ and $x \in N(L_A - \lambda I_{F^n})$.

Determining eigenvectors and eigenvalues of a lin. operator

Let V be a v.s., $\dim(V) = n$. Let β be an ordered basis for V.
Let $T \in \mathcal{L}(V)$ be a lin. operator on V.
Summarizing the results of the previous section, we describe how to determine the e.val's and the e.vec's of T.

1) Determine the matrix representation $[T]_\beta$ of T.
2) Determine the e.val's of T.
   $\lambda \in F$ is an e.val of $T \iff \lambda$ is a root of the char. polynomial of T.
   That is, we need to find the solutions $x \in F$ of $\det([T]_\beta - x I_n) = 0$.
   There are at most n distinct solutions $\lambda_1, \ldots, \lambda_n$.
3) Now for each e.val. λ of T, we can determine the corresponding e.vec's. We have:
   $T(v) = \lambda v \iff (T - \lambda I_V)(v) = 0 \iff [T - \lambda I_V]_\beta [v]_\beta = 0$.
   Therefore, eigenvectors corresponding to λ are the solutions of this system of linear equations. (more precisely, solving this system we find the β-coordinates $[v]_\beta$, which then determines v).