Isomorphisms.

Sometimes two vector spaces may consist of objects of very different nature, but behave identically from the algebraic point of view. We describe a precise way of "identifying" vector spaces with each other.

Definition. Let V, W be v.s. We say that V is isomorphic to W if there exists a lin. transf. $T: V \to W$ that is invertible.
Such a lin. transf. is called an isomorphism from V onto W.

Note. 1) V is isomorphic to V (using I_V).
2) V is isomorphic to W $\Leftrightarrow$ W is isomorphic to V.
3) If V is isomorphic to W and W is isomorphic to Z, then V is isomorphic to Z. Exersice.
Thus isomorphism is an equivalence relation on vector spaces.

Example. Let $T: F^2 \to P_1(F)$ be given by $T(a_1, a_2) = a_1 + a_2 x$.
Then T is an isomorphism, so F^2 is isomorphic to $P_1(F)$.

Theorem 2.19. Let V, W be fin. dim. v.s. over F.
Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Proof. "$\Rightarrow$" Let $T: V \to W$ be an isomorphism from V to W.
By the lemma above, $\dim(V) = \dim(W)$.

"$\Leftarrow$" Suppose $\dim(V) = \dim(W)$, and let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_n\}$ be bases for V and W, resp.
By Theorem 2.6, there exists $T: V \to W$ s.t. T is lin. and $T(v_i) = w_i$ for $i = 1, \dots, n$.
By Theorem 2.2,
$R(T) = \operatorname{Span}(T(\beta)) = \operatorname{Span}(\gamma) = W$, so T is surjective.
By Theorem 2.5, T is also injective.
Hence T is an isomorphism.

Corollary. Let V be a v.s. over F.
Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Up to this point, we have associated linear transformations with their matrix representations, and we have seen many analogies between the operations on $\mathcal{L}(V,W)$ and $M_{m \times n}(F)$.
Now we can show that these two spaces may be identified.

Theorem 2.20.
Let V, W be v.s. over F, $\dim(V) = n$, $\dim(W) = m$.
Let β, γ be ordered bases for V and W, respectively.
Then the function $\Phi: \mathcal{L}(V,W) \to M_{m \times n}(F)$ defined by
$$\Phi(T) = [T]_\beta^\gamma \quad \text{for all } T \in \mathcal{L}(V,W)$$
is an isomorphism.

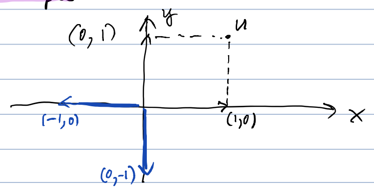
Proof.
By Theorem 2.8, Φ is linear. So remains to show Φ is a bijection.
That is, we need to show that for every $A \in M_{m \times n}(F)$, there is a <u>unique</u> lin. transf. $T: V \to W$ s.t. $\Phi(T) = A$.
Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$, and let $A \in M_{m \times n}(F)$ be given.
By Theorem 2.6, there exists a unique lin. transf. $T: V \to W$ s.t.
$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i \quad \text{for } 1 \le j \le n.$$
But this means that $[T]_\beta^\gamma = A$, or $\Phi(T) = A$. Thus Φ is an isomorphism.

Corollary. If $\dim(V)=n$, $\dim(W)=m$, then $\dim(\mathcal{L}(V,W))=mn$.
(by the previous theorem, as $\dim(M_{m\times n}(F))=mn$).

Change of coordinate matrix

We have seen that once we fix an ordered basis β of a v.s. V to every vector $u\in V$ we can assign its coordinates $[u]_\beta$. And similarly, for $T:V\to V$, we assign its matrix rep. $[T]_\beta$.
Hower, these coordinates depend on β! And can be different for another choice of an ordered basis.

Example.



$V=\mathbb{R}^2$
$\beta=\{(1,0),(0,1)\}$ – ordered basis
$[u]_\beta=\begin{pmatrix}1\\1\end{pmatrix}$
$\gamma=\{(-1,0),(0,-1)\}$ – another ordered basis for V.
$[u]_\gamma=\begin{pmatrix}-1\\-1\end{pmatrix}$.

We would like a method to calculate $[u]_\gamma$ from $[u]_\beta$, for an arbitrary choice of β and γ.

Definition. Let β and β' be two ordered bases for a fin. dim. v.s. V.
We define the change of coordinate matrix (or "change of basis matrix") to be $Q=[I_V]_{\beta'}^{\beta}$.

Theorem 2.22.
1) Q is invertible. (and $Q^{-1}=[I_V]_{\beta}^{\beta'}$).
2) For any $v\in V$, $[v]_\beta=Q[v]_{\beta'}$.

Proof.
1) As I_V is invertible, Q is also invertible by Thm 2.18
2) For any $v\in V$,
$[v]_\beta=[I_V(v)]_\beta=[I_V]_{\beta'}^{\beta}[v]_{\beta'}=Q[v]_{\beta'}$, by Theorem 2.14.

So, multiplying by Q changes the β'-coordinates of a vector into its β-coordinates.
And multiplying by Q^{-1} changes β-coordinates into β'-coordinates.

Example.
In the example above, $[I_V]_\gamma^\beta=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$.
And $[u]_\gamma=\begin{pmatrix}-1\\-1\end{pmatrix}$.
Hence $[u]_\beta=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$.

Definition A lin. transf. $T:V\to V$ from a v.s. V to itself is called a linear operator on V.
• Now we determine how to calculate $[T]_\beta$ from $[T]_{\beta'}$, for β,β' two ordered bases for V.

Theorem 2.23. Let T be a lin. operator on a fin. dim. v.s. V.
Let β,β' be ordered bases for V.
Let $Q=[I_V]_{\beta'}^{\beta}$ be the change of coordinate matrix, changing β'-coord's into β-coords.
Then
$[T]_{\beta'}=Q^{-1}[T]_\beta Q$.

Proof.
Recall that $T=I_VT=TI_V$.
$Q[T]_{\beta'}=[I_V]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}=[I_VT]_{\beta'}^{\beta}=[TI_V]_{\beta'}^{\beta}=[T]_\beta^\beta[I_V]_{\beta'}^{\beta}=[T]_\beta Q$. (by Theorem 2.11).
Therefore
$[T]_{\beta'}=Q^{-1}[T]_\beta Q$.

Example.

Consider the lin. operator T on $\mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$.

Let $\beta = \{(1,0),(0,1)\}$ and $\beta' = \{(-1,0),(0,-1)\}$ be ordered bases.

By the previous example:

$Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Also $Q^{-1} = [I_V]_{\beta}^{\beta'} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Also $[T]_\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Hence $[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Determinants

Definition

Let $A \in M_{n\times n}(F)$.

1) For any $1 \le i,j \le n$ we define the cofactor matrix of the entry of A in row i and column j to be the matrix $\tilde{A}_{ij} \in M_{(n-1)\times(n-1)}(F)$ obtained from A by deleting row i and column j.

2) The determinant of A, denoted $\det(A)$, is a scalar in F defined recursively as follows:
   - if $n=1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$.
   - for $n \ge 2$, we define
   $$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}) \quad \text{for any } 1 \le i \le n$$
   (this formula gives the same value for any i! See Theorem 4.4).

3) Equivalently, we have
$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij}) \quad \text{for any } 1 \le j \le n.$$

Example. Let's consider the case $n=2$.

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_{2\times 2}(F)$ be given.

According to the definition, we can evaluate its determinant along any row i.

Let's take $i=1$.

Then the cofactor matrices are $\tilde{A}_{1,1} = (A_{22})$ and $\tilde{A}_{1,2} = (A_{21})$.

So $\det(\tilde{A}_{1,1}) = A_{22}$, $\det(\tilde{A}_{1,2}) = A_{21}$ and

$$\det(A) = \sum_{j=1}^{2} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1,j}) = A_{11} \cdot A_{22} - A_{12} A_{21}.$$ – the familiar formula.

Example.

Let $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$.

Again, let's calculate $\det(A)$ using cofactors along the 1^{st} row. We obtain:

$$\det(A) = (-1)^{1+1} A_{11} \det(\tilde{A}_{1,1}) + (-1)^{1+2} A_{12} \cdot \det(\tilde{A}_{1,2}) + (-1)^{1+3} A_{13} \det(\tilde{A}_{1,3}) =$$

$$= (-1)^2 \cdot 1 \cdot \det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3 \cdot 3 \cdot \det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + (-1)^4 \cdot (-3) \cdot \det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} =$$

$$= 1 \cdot (-5 \cdot (-6) - 2 \cdot 4) - 3(-3 \cdot (-6) - 2 \cdot (-4)) - 3(-3 \cdot 4 - (-5) \cdot (-4)) =$$

$$= 1 \cdot 22 - 3 \cdot 26 - 3 \cdot (-32) = 40.$$

Properties of the determinant (See Sections 4.2–4.4 in the textbook for the proofs)

Let $A \in M_{n\times n}(F)$. If B is a matrix obtained from A by

1) switching two rows (or two columns), then
$\det(B) = -\det(A)$.
2) multiplying a row (or a column) of A by a scalar $c \in F$, then
$\det(B) = c \cdot \det(A)$.
3) adding a multiple of row i to row j (or a multiple of column i to column j), then
$\det(B) = \det(A)$.

These properties are helpful for computing determinants.
We also have the following properties:

4) If $B \in M_{n\times n}(F)$, then
$\det(AB) = \det(A) \cdot \det(B)$,
5) A is invertible if and only if $\det(A) \neq 0$. Furthermore,
$$\det(A^{-1}) = \frac{1}{\det(A)}.$$
6) If $I_n \in M_{n\times n}(F)$ is the identity matrix, then
$\det(I_n) = 1$.
7) $\det(A) = \det(A^t)$.

The operations on the rows of a matrix described in 1), 2) and 3) above are called **elementary row operations**.

Fact. Using these operations, we can transform any square matrix into an **upper triangular matrix**. That is, a matrix of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ 0 & & \ddots & \\ & & & A_{nn} \end{pmatrix}$$ – all entries below the diagonal are 0.

Fact. If $A \in M_{n\times n}(F)$ is upper triangular, then $\det(A) = A_{11} \cdot A_{22} \cdot \ldots \cdot A_{nn}$.

These two facts simplify calculating the determinants.

Example.
Let $B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$.

Applying elementary rows operations, we have

$$B \xrightarrow{(1)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}.$$

exchanging rows 1 and 2 adding $2\times(\text{row } 1)$ to row 3 adding $10\times(\text{row } 2)$ to row 3

As (3) doesn't change the determinant, we have

$\det\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} = -2\cdot 1\cdot(24) = -48$, and as (1) only changes the sign of det, we have $\det(B) = 48$.