On the number of Dedekind cuts

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ded κ

- Let κ be an *infinite* cardinal.

Definition

$\operatorname{ded} \kappa = \sup\{|I| \colon I \text{ is a linear order with a dense subset of size} \leq \kappa\}$.

- In general the supremum need not be attained.
- In model theory this function arises naturally when one wants to count types.

Equivalent ways to compute

The following cardinals are the same:

1. $\operatorname{ded}\kappa$,
2. $\sup\{\lambda$: exists a linear order I of size $\leq \kappa$ with λ Dedekind cuts$\}$,
3. $\sup\{\lambda$: exists a regular μ and a linear order of size $\leq \kappa$ with λ cuts of *cofinality* μ *on both sides*$\}$
   (by a theorem of Kramer, Shelah, Tent and Thomas),
4. $\sup\{\lambda$: exists a regular μ and a tree T of size $\leq \kappa$ with λ branches of length $\mu\}$.

Some basic properties of $\operatorname{ded}\kappa$

- $\kappa < \operatorname{ded}\kappa \leq 2^\kappa$ for every infinite κ
  (for the first inequality, let μ be minimal such that $2^\mu > \kappa$, and consider the tree $2^{<\mu}$)
- $\operatorname{ded}\aleph_0 = 2^{\aleph_0}$
  (as $\mathbb{Q} \subseteq \mathbb{R}$ is dense)
- Assuming GCH, $\operatorname{ded}\kappa = 2^\kappa$ for all κ.
- [Baumgartner] If $2^\kappa = \kappa^{+n}$ (i.e. the nth sucessor of κ) for some $n \in \omega$, then $\operatorname{ded}\kappa = 2^\kappa$.
- So is $\operatorname{ded}\kappa$ the same as 2^κ in general?

Fact

[Mitchell] For any κ with $\operatorname{cf}\kappa > \aleph_0$ it is consistent with ZFC that $\operatorname{ded}\kappa < 2^\kappa$.

Counting types

- Let T be an arbitrary complete first-order theory in a countable language L.
- For a model M, $S_T(M)$ denotes the space of types over M (i.e. the space of ultrafilters on the boolean algebra of definable subsets of M).
- We define $f_T(\kappa) = \sup\{|S_T(M)| : M \models T, |M| = \kappa\}$.

Fact

[Keisler], [Shelah] For any countable T, f_T is one of the following functions: κ, $\kappa + 2^{\aleph_0}$, $\kappa^{\aleph_0}$, $\operatorname{ded}\kappa$, $(\operatorname{ded}\kappa)^{\aleph_0}$, 2^κ (and each of these functions occurs for some T).

- These functions are distinguished by combinatorial dividing lines of Shelah, resp. ω-stability, superstability, stability, non-multi-order, NIP (more later).

Further properties of ded κ

- So we have $\kappa < \operatorname{ded}\kappa \leq (\operatorname{ded}\kappa)^{\aleph_0} \leq 2^{\aleph_0}$ and $\operatorname{ded}\kappa = 2^\kappa$ under GCH.
- [Keisler, 1976] Is it consistent that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$?

Theorem (*)

[Ch., Kaplan, Shelah] It is consistent with ZFC *that* $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0}$ *for some* κ.

- Our proof uses Easton forcing and elaborates on Mitchell's argument. We show that e.g. consistently $\operatorname{ded}\aleph_\omega = \aleph_{\omega+\omega}$ and $(\operatorname{ded}\aleph_\omega)^{\aleph_0} = \aleph_{\omega+\omega+1}$.
- **Problem**. Is it consistent that $\operatorname{ded}\kappa < (\operatorname{ded}\kappa)^{\aleph_0} < 2^\kappa$ at the same time for some κ.

Bounding exponent in terms of $\operatorname{ded}\kappa$

- Recall that by Mitchell consistently $\operatorname{ded}\kappa < 2^{\kappa}$. However:

Theorem (**)

[Ch., Shelah] $2^{\kappa} \leq \operatorname{ded}(\operatorname{ded}(\operatorname{ded}(\operatorname{ded}\kappa)))$ *for all infinite* κ.

- The proof uses Shelah's PCF theory.
- **Problem**. What is the minimal number of iterations which works for all models of ZFC? At least 2, and 4 is enough.

Two-cardinal models

- As always, T is a first-order theory in a countable language L, and let $P(x)$ be a predicate from L.
- For cardinals $\kappa \geq \lambda$ we say that $M \models T$ is a (κ, λ)-model if $|M| = \kappa$ and $|P(M)| = \lambda$.
- A classical question is to determine implications between existence of two-cardinal models for different pairs of cardinals (Vaught, Chang, Morley, Shelah, ...).

Arbitrary large gaps

Fact

[Vaught] Assume that for some κ*,* T *admits a* $(\beth_n(\kappa), \kappa)$*-model for all* $n \in \omega$*. Then* T *admits a* (κ', λ')*-model for any* $\kappa' \geq \lambda'$*.*

Example

Vaught's theorem is optimal. Fix $n \in \omega$, and consider a structure M in the language $L = \{P_0(x), \ldots, P_n(x), \in_0, \ldots, \in_{n-1}\}$ in which $P_0(M) = \omega$, $P_{i+1}(M)$ is the set of subsets of $P_i(M)$, and $\in_i \subseteq P_i \times P_{i+1}$ is the belonging relation. Let $T = \mathrm{Th}(M)$. Then M is a $(\beth_n, \aleph_0)$-model of T, but it is easy to see by "extensionality" that for any $M' \models T$ we have $|M'| \leq \beth_n(|P_0(M')|)$.

- However, the theory in the example is wild from the model theoretic point of view, and stronger transfer principles hold for tame classes of theories.

Two-cardinal transfer for "tame" classes of theories

- A theory is *stable* if $f_T(\kappa) \leq \kappa^{\aleph_0}$ for all κ. Examples: $(\mathbb{C}, +, \times, 0, 1)$, equivalence relations, abelian groups, free groups, planar graphs, ...

Fact

[Lachlan], [Shelah] If T is stable and admits a (κ, λ)-model for some $\kappa > \lambda$, then it admits a (κ', λ')-model for any $\kappa' \geq \lambda'$.

- A theory is *o*-minimal if every definable set is a finite union of points and intervals with respect to a fixed definable linear order (e.g. $(\mathbb{R}, +, \times, 0, 1, \exp)$).

Fact

[T. Bays] If T is o-minimal and admits a (κ, λ)-model for some $\kappa > \lambda$, then it admits a (κ', λ')-model for any $\kappa' \geq \lambda'$.

NIP theories

Definition

A theory is NIP (No Independence Property) if it cannot encode subsets of an infinite set. That is, there are **no** model $M \models T$, tuples $(a_i)_{i \in \omega}, (b_s)_{s \subseteq \omega}$ and formula $\phi(x, y)$ such that $M \models \phi(a_i, b_s)$ holds if and only if $i \in s$.

- Equivalently, uniform families of definable sets have finite VC-dimension.

Fact

[Shelah] T is NIP *if and only if* $f_T(\kappa) \leq (\operatorname{ded} \kappa)^{\aleph_0}$ *for all* κ*.*

Example

The following theories are NIP:

- Stable theories,
- *o*-minimal theories,
- colored linear orders, trees, algebraically closed valued fields, *p*-adics.

Vaught's bound is optimal for NIP

- So can one get a better bound in Vaught's theorem restricting to NIP theories?

Theorem (***)

[Ch., Shelah] For every $n \in \omega$ there is an NIP theory T which admits a $(\beth_n, \aleph_0)$-model, but no $(\beth_\omega, \aleph_0)$-models.

Proof.

1. Consider $T = \mathrm{Th}(\mathbb{R}, \mathbb{Q}, <)$ with $P(x)$ naming $\mathbb{Q}$, it is NIP. Then T admits a $(2^{\aleph_0}, \aleph_0)$-model, but for every $M \models T$ we have $|M| \leq \mathrm{ded}(|P(M)|)$, as $P(M)$ is dense in M. The idea is to iterate this construction.
2. Picture.
3. Doing this generically, we can ensure that T eliminates quantifiers and is NIP. In n steps we get a $(\mathrm{ded}^n \aleph_0, \aleph_0)$-model. Applying Theorem (**) we see that in $4n$ steps we get a $(\beth_n, \aleph_0)$-model, but of course no $(\beth_\omega, \aleph_0)$-models.

Comments

- Elaborating on the same technique we can show that the Hanf number for omitting a type is as large in NIP theories as in arbitrary theories (again unlike the stable and the *o*-minimal cases where it is much smaller).
- **Problem**. Transfer between cardinals close to each other. Let T be NIP and assume that it admits a (κ, λ)-model for some $\kappa > \lambda$. Does it imply that it admits a (κ', λ)-model for all $\lambda \leq \kappa' \leq \operatorname{ded} \lambda$?
- **Conjecture**. There is a better bound in the finite dp-rank case (connected to the existence of an indiscernible subsequence in every sufficiently long sequence).

Tree exponent

Definition

For two cardinals λ and μ, let
$\lambda^{\mu,\mathrm{tr}} = \sup\{\kappa\colon$ there is a tree T with λ many nodes and κ branches of length $\mu\}$.

- Note that $\kappa^{\kappa,\mathrm{tr}} = \operatorname{ded}\kappa$.

Finer counting of types

- Let $\kappa \geq \lambda$ be infinite cardinals, T a complete countable theory as always.

Definition

$g_T(\kappa, \lambda) = \sup\{|P|:$ P is a family of pairwise-contradictory partial types, each of size $\leq \kappa$, over some A with $|A| \leq \lambda\}$.

- Note that $g_T(\kappa, \kappa) = f_T(\kappa)$.
- **Conjecture**. There are finitely many possibilities for g_T.

Theorem

[Ch., Shelah] True assuming GCH *or assuming* $\lambda \gg \kappa$.

- The remaining problem: show that if T is NIP then $g_T(\kappa, \lambda) \leq \lambda^{\kappa,\mathrm{tr}}$.

Some comments

1. T is ω-stable $\Rightarrow g_T(\kappa, \lambda) = \lambda$ for all $\lambda \geq \kappa \geq \aleph_0$.
2. T is superstable, not ω-stable $\Rightarrow g_T(\kappa, \lambda) = \lambda + 2^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$.
3. T is stable, not superstable $\Rightarrow g_T(\kappa, \lambda) = \lambda^{\aleph_0}$ for all $\lambda \geq \kappa \geq \aleph_0$.
4. T is supersimple, unstable $\Rightarrow g_T(\kappa, \lambda) = \lambda + 2^{\kappa}$ for all $\lambda \geq \kappa \geq \aleph_0$.
5. T is simple, not supersimple $\Rightarrow g_T(\kappa, \lambda) = \lambda^{\aleph_0} + 2^{\kappa}$ for all $\lambda \geq \kappa \geq \aleph_0$.
6. T is not simple, not NIP $\Rightarrow g_T(\kappa, \lambda) = \lambda^{\kappa}$ for all $\lambda \geq \kappa \geq \aleph_0$.
7. T is NIP, not simple:
    - $g_T(\kappa, \lambda) = \lambda^{\kappa}$ for $\lambda^{\kappa} > \lambda + 2^{\kappa}$ (by set theory),
    - for $\lambda \leq 2^{\kappa}$ we have $g_T(\kappa, \lambda) \geq \lambda^{\kappa,\mathrm{tr}}$. **So if** ded $\kappa = 2^{\kappa}$ **then we are done.**

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