# Recognizing groups and fields in Erdős geometry and model theory

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# Hypergraphs and Zarankiewicz's problem

- We fix $r \in \mathbb{N}_{\geq 2}$ and let $H = (V_1, \ldots, V_r; E)$ be an $r$-partite and $r$-uniform hypergraph (or just *$r$-hypergraph*) with vertex sets $V_1, \ldots, V_r$ with $|V_i| = n_i$, (hyper-) edge set $E \subseteq \prod_{i \in [r]} V_i$, and $n = \sum_{i=1}^{r} n_i$ is the total number of vertices.
- When $r = 2$, we say "bipartite graph" instead of "2-hypergraph".
- For $k \in \mathbb{N}$, let $K_{k,\ldots,k}$ denote the complete $r$-hypergraph with each part of size $k$ (i.e. $V_i = [k]$ and $E = \prod_{i \in [k]} V_i$).
- $H$ is *$K_{k,\ldots,k}$-free* if it does note contain an isomorphic copy of $K_{k,\ldots,k}$.
- Zarankiewicz's problem: for fixed $r, k$, what is the maximal number of edges $|E|$ in a $K_{k,\ldots,k}$-free $r$-hypergraph $H$? (As a functions of $n_1, \ldots, n_r$.)

# Number of edges in a $K_{k,\dots,k}$-free hypergraph

- The following fact is due to [Kővári, Sós, Turán'54] for $r = 2$ and [Erdős'64] for general $r$.

Fact (The Basic Bound)

*If $H$ is a $K_{k,\dots,k}$-free $r$-hypergraph then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$.*

- So the exponent is slightly better than the maximal possible $r$ (we have $n^r$ edges in $K_{n,\dots,n}$). A probabilistic construction in [Erdős'64] shows that this bound cannot be substantially improved (but whether it is sharp up to a constant is widely open).
- Restricting to hypergraphs that are defined "geometrically", one might expect stronger bounds on the exponent.

# Semialgebraic hypergraphs

- A set $X \subseteq \mathbb{R}^d$ is *semialgebraic* if $X$ is a finite union of sets of the form

$$\left\{ \bar{x} \in \mathbb{R}^d : f_1(\bar{x}) \geq 0, \ldots, f_p(\bar{x}) \geq 0, f_{p+1}(\bar{x}) > 0, \ldots, f_q(\bar{x}) > 0 \right\},$$

  where $p \leq q \in \mathbb{N}$ and each $f_i \in \mathbb{R}[\bar{x}]$ is a polynomial in $d$ variables.
- $X$ has *(description) complexity* $t$ if $d \leq t$, it is a union of at most $t$ such sets, $q \leq t$ and $\deg(f_i) \leq t$ for all $i$.
- A finite $r$-hypergraph $H = (V_1, \ldots, V_r; E)$ is *semialgebraic, of complexity* $t$ if $V_i \subseteq \mathbb{R}^{d_i}$ for some $d_i$ and $E = \left(\prod_{i \in [r]} V_i\right) \cap X$ for some semialgebraic set $X \subseteq \mathbb{R}^{d_1 + \ldots + d_r}$ of complexity $t$ (up to isomorphism).
- A lot of (hyper-)graphs arising in incidence combinatorics of elementary geometric shapes are semialgebraic, of small complexity.

## Example: point-line incidences on the plane

- Let $I \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the incidence relation between points and lines on the plane, i.e.

$$I(x_1, x_2; y_1, y_2) \iff x_2 = y_1 x_1 + y_2.$$

- Then $I$ is semialgebraic (of complexity 2) and $K_{2,2}$-free (for any two points belong to at most one line, and vice versa).
- Let $V_1$ be a set of $n_1$ points and $V_2$ a set of $n_2$ lines on the plane $\mathbb{R}^2$, and $E := I \restriction_{V_1 \times V_2}$. Then the bipartite graph $(V_1, V_2; E)$ satisfies the basic bound of Kővári, Sós, Turán:

$$|E| = O\left(n^{\frac{3}{2}}\right).$$

- While this is optimal for general graphs, utilizing the geometry of the reals:

### Fact (Szémeredi-Trotter '83)

*In fact,* $|E| = O\left(n^{\frac{4}{3}}\right)$.

- Note that $\frac{4}{3} < \frac{3}{2}$.

# Zarankiewicz for semialgebraic (hyper-)graphs

- Szémeredi-Trotter theorem has numerous generalizations for semialgebraic graphs, e.g. [Pach, Sharir'98], [Elekes, Szabó'12], and more generally

### Fact (Fox, Pach, Sheffer, Suk, Zahl'17)

*If $(V_1, V_2; E)$, with $V_i \subseteq \mathbb{R}^{d_i}$, is a semialgebraic bipartite graph of complexity $t$ and $K_{k,k}$-free, then for any $\varepsilon > 0$,*

$$|E| = O_{t,d_1,d_2,k,\varepsilon}\left(n_1^{\frac{d_2(d_1-1)}{d_1 d_2 - 1}+\varepsilon} n_2^{\frac{d_1(d_2-1)}{d_1 d_2 - 1}} + n_1 + n_2\right).$$

- Generalizations to semialgebraic hypergraphs [Do'18].
- Moral: for semialgebraic, the bound is of the form $O(n^{e-\varepsilon})$, where $e$ is given by the basic bound for arbitrary graphs.

# Connections to the “trichotomy principle” in model theory

- The *trichotomy principle* in model theory: in a sufficiently tame context (including semialgebraic), every structure is either “trivial”, or essentially a vector space, or interprets a field (see below).
- In this talk: the exponents in Zarankiewicz bounds for semialgebraic (hyper-)graphs appear to reflect the trichotomy principle, and detect presence of algebraic structures (groups, fields).
- Instances of this principle are also known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures.

# Elekes-Szabó theorem, 1

- [Erdős, Szemerédi'83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,

$$\max \{|A + A|, |A \cdot A|\} = \Omega \left( |A|^{1+c} \right).$$

- [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any $\varepsilon$).
- [Elekes, Rónyai'00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree $d$, then for all $A, B \subseteq_n \mathbb{R}$,

$$|f(A \times B)| = \Omega_d \left( n^{\frac{4}{3}} \right),$$

unless $f$ is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials $g, h, i$.

# Elekes-Szabó theorem, 2

- [Elekes-Szabó'12] provide a conceptual generalization: for any algebraic surface $Q(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:
  1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq_n \mathbb{R}$ we have
  $$|Q \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}).$$
  2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \to V$ such that
  $$\pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \Leftrightarrow Q(x_1, x_2, x_3)$$
  for all $x_i \in U_i$.

# Generalizations of the Elekes-Szabó theorem

Let $Q \subseteq X_1 \times \ldots \times X_r$ be an algebraic surface with finite-to-one projection onto any $r-1$ coordinates and $\dim(X_i) = k$.

1. [Elekes, Szabó'12] $r = 3$, $k$ arbitrary over $\mathbb{C}$ (only count on grids in *general position*, correspondence with a complex algebraic group of dimension $k$);
2. [Raz, Sharir, de Zeeuw'18] $r = 4$, $k = 1$ over $\mathbb{C}$;
3. [Raz, Shem-Tov'18] $k = 1$, $Q$ of the form $f(x_1, ..., x_{r-1}) = x_r$ for any $r$ over $\mathbb{C}$.
4. [Bays, Breuillard'18] $r$ and $k$ arbitrary over $\mathbb{C}$, recognized that the arising groups are abelian (however no bounds on $\gamma$);
5. Related work: [Raz, Sharir, de Zeeuw'15], [Wang'15]; [Bukh, Tsimmerman' 12], [Tao'12]; [Hrushovski'13]; [Jing, Roy, Tran'19].
6. [C., Peterzil, Starchenko] Any $s$ and $k$, over $\mathbb{R}$, $\mathbb{C}$ (and much more) and explicit bounds on $\gamma$. A special case:

### Theorem (C., Peterzil, Starchenko)

*Assume $r \geq 3$ and $Q \subseteq \mathbb{R}^r$ is semi-algebraic, of description complexity $D$, such that the projection of $Q$ to any $r-1$ coordinates is finite-to-one. Then exactly one of the following holds.*

1. *There exists a constant $c$, depending only on $r$ and $D$, such that: for any finite $A_i \subseteq_n \mathbb{R}$, $i \in [r]$, we have*

$$|Q \cap (A_1 \times \ldots \times A_r)| = O_{r,d}\left(n^{r-1-\gamma}\right),$$

   *where $\gamma = \frac{1}{3}$ if $r \geq 4$, and $\gamma = \frac{1}{6}$ if $r = 3$.*

2. *There exist open sets $U_i \subseteq \mathbb{R}, i \in [r]$, an open set $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \to V$ such that*

$$\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \Leftrightarrow Q(x_1, \ldots, x_r)$$

   *for all $x_i \in U_i, i \in [r]$.*

# Remarks

1. In general, for semialgebraic $Q \subseteq X_1 \times \ldots \times X_r$ with $\dim(X_i) = k$, holds with $V$ a neighborhood of 0 in an abelian Lie group of dimension $k$.
2. In fact, our theorem is for $Q$ definable in an arbitrary $o$-minimal expansion of $\mathbb{R}$ — so $Q$ can be defined not only using polynomial (in-)equalities, but also using $e^x$ and restricted analytic functions.
3. One ingredient — improved Zarankiewicz bounds also hold in this context ([Basu, Raz], [C., Galvin, Starchenko]).
4. Another – a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a "generic chunk". We discuss a simple purely combinatorial case:

# Recognizing groups, 1

1. Assume that $(G, +, 0)$ is an abelian group, and consider the $r$-ary relation $Q \subseteq \prod_{i \in [r]} G$ given by $x_1 + \ldots + x_r = 0$.
2. Then $Q$ is easily seen to satisfy the following two properties, for any permutation of the variables of $Q$:

$$\forall x_1, \ldots, \forall x_{r-1} \exists! x_r Q(x_1, \ldots, x_r), \tag{P1}$$

$$\forall x_1, x_2 \forall y_3, \ldots y_r \forall y_3', \ldots, y_r' \Big( Q(\bar{x}, \bar{y}) \wedge Q(\bar{x}, \bar{y}') \rightarrow \tag{P2}$$
$$\big(\forall x_1', x_2' Q(\bar{x}', \bar{y}) \leftrightarrow Q(\bar{x}', \bar{y}')\big)\Big).$$

We show a converse, assuming $r \geq 4$:

# Recognizing groups, 2

## Theorem (C., Peterzil, Starchenko)

*Assume $r \in \mathbb{N}_{\geq 4}$, $X_1, \ldots, X_r$ and $Q \subseteq \prod_{i \in [r]} X_i$ are sets, so that $Q$ satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group $(G, +, 0_G)$ and bijections $\pi_i : X_i \to G$ such that for every $(a_1, \ldots, a_r) \in \prod_{i \in [r]} X_i$ we have*

$$Q(a_1, \ldots, a_r) \iff \pi_1(a_1) + \ldots + \pi_r(a_r) = 0_G.$$

- If $X_1 = \ldots = X_r$, property (P1) is equivalent to saying that the relation $Q$ is an $(r-1)$-*dimensional permutation* on the set $X_1$, or a *Latin* $(r-1)$-*hypercube*, as studied by Linial and Luria. Thus the condition (P2) characterizes, for $r \geq 3$, those Latin $r$-hypercubes that are given by the relation "$x_1 + \ldots + x_{r-1} = x_r$" in an abelian group.

# Recognizing fields

- For the semialgebraic $K_{2,2}$-free point-line incidence relation $Q = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1 x_1 + y_2\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have the (optimal) lower bound $|Q \cap (V_1 \times V_2)| = \Omega(n^{\frac{4}{3}})$.
- To define it we use both addition and multiplication, i.e. the field structure.
- This is not a coincidence — any non-trivial lower bound on the Zarankiewicz's exponent of $Q$ allows to recover a field from it:

**Theorem (Basit, C., Starchenko, Tao, Tran)**

*Assume that $Q \subseteq \mathbb{R}^d = \prod_{i \in [r]} \mathbb{R}^{d_i}$ for some $r, d_i \in \mathbb{N}$ is semialgebraic and $K_{k,\ldots,k}$-free, but $|Q \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})$. Then a real closed field is definable in the first-order structure $(\mathbb{R}, <, Q)$.*

# Ingredients

- An almost optimal Zarankiewicz bound for *semilinear* hypergraphs.
- The trichotomy theorem for *o*-minimal structures from model theory [Peterzil, Starchenko].

# Semilinear relations of bounded complexity

- A set $X \subseteq \mathbb{R}^d$ is *semilinear*, of complexity $t$, if $X$ is a union of at most $t$ sets of the form

  $$\left\{\bar{x} \in \mathbb{R}^d : f_1(\bar{x}) \leq 0, \ldots, f_p(\bar{x}) \leq 0, f_{p+1}(\bar{x}) < 0, \ldots, f_q(\bar{x}) < 0\right\},$$

  where $p \leq q \leq t \in \mathbb{N}$ and each $f_i : \mathbb{R}^d \to \mathbb{R}$ is a *linear* function

  $$f(x_1, \ldots, x_d) = \lambda_1 x_1 + \ldots + \lambda_d x_d + a$$

  for some $\lambda_i, a \in \mathbb{R}$.

# Zarankiewicz bound for relations of bounded box complexity

## Theorem (BCSTT)

*For any integers $r \geq 2, s \geq 0, k \geq 2$ there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = \beta(r, s) \in \mathbb{N}$ such that: for any finite $K_{k,\ldots,k}$-free semilinear $r$-hypergraph $H = (V_1, \ldots, V_r; E)$ with $E \subseteq \prod_{i \in [r]} V_i$ of complexity $\leq s$ we have*

$$|E| \leq \alpha n^{r-1} (\log n)^{\beta} .$$

*Moreover, we can take $\beta(r, s) := s(2^{r-1} - 1)$.*

- In particular, $|E| = O_{r,s,k,\varepsilon}(n^{r-1+\varepsilon})$ for any $\varepsilon > 0$.

# An application to incidences with polytopes

- Applying with $r = 2$ we get the following:

## Corollary (BCSTT)

*For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.*
*Let $d \in \mathbb{N}$ and $H_1, \ldots, H_q \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_q$.*
*For any set $V_1$ of $n_1$ points in $\mathbb{R}^d$ and any set $V_2$ of $n_2$ polytopes in $\mathcal{F}$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $\alpha n (\log n)^q$ incidences.*

- In particular (a similar result was obtained independently by [Tomon, Zakharov]):

## Corollary (BCSTT)

*For any set $V_1$ of $n_1$ points and any set $V_2$ of $n_2$ (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k}\left(n(\log n)^{2d}\right)$ incidences.*

# Dyadic rectangles and a lower bound

- Is the logarithmic factor necessary?
- We focus on the simplest case of incidences with rectangles with axis-parallel sides in $\mathbb{R}^2$. The previous corollary gives the bound $O_{d,k}\left(n(\log n)^4\right)$.
- A box is *dyadic* if it is the direct products of intervals of the form $[s2^t, (s+1)2^t)$ for some integers $s, t$.
- Using a different argument, restricting to dyadic boxes we get a stronger upper bound $O\left(n\frac{\log n_1}{\log\log n_1}\right)$, and give a construction showing a matching lower bound (up to a constant).

## Problem

*What is the optimal bound on the power of* $\log n$*? In particular, does it have to grow with the dimension* $d$*?*

# Geometric weakly locally modular theories

- In our bounds, we can get rid of the logarithmic factor entirely restricting to the family of all finite $r$-hypergraphs induced by a given $K_{k,\dots,k}$-free relation (as opposed to all $K_{k,\dots,k}$-free $r$-hypergraphs induced by a given relation).
- A first-order structure is *geometric* if the algebraic closure operator satisfies the *Exchange Principle* and the quantifier $\exists^\infty$ is eliminated.
- Hence, in a model of a geometric theory, acl defines a well-behaved notion of independence $\perp$ (equivalently, a matroid).
- A geometric structure is *(weakly) locally modular* if for any small subsets $A, B$ there exists some small subset $C \perp_\emptyset AB$ such that $A \perp_{\mathrm{acl}(AC) \cap \mathrm{acl}(BC)} B$.
- Moral: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.

# Recovering a field in the *o*-minimal case

## Fact (Peterzil, Starchenko'98)

*Let $\mathcal{M}$ be an o-minimal saturated structure. TFAE:*

- *$\mathcal{M}$ is not locally modular;*
- *there exists a real closed field definable in $\mathcal{M}$.*

- [Marker, Peterzil, Pillay'92] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\cdot \restriction_{[0,1]^2}$ is definable in $(\mathbb{R}, <, +, X)$. In particular, it is not locally modular.
- Combining all of this, we get the result.

# Thank you!

- *Model-theoretic Elekes-Szabó for stable and o-minimal hypergraphs*, Artem Chernikov, Ya'acov Peterzil, Sergei Starchenko (arXiv:2104.02235)
- *Zarankiewicz's problem for semilinear hypergraphs*, Artem Chernikov, Abdul Basit, Sergei Starchenko, Terence Tao and Chieu-Minh Tran (arXiv:2009.02922)