NTP_1

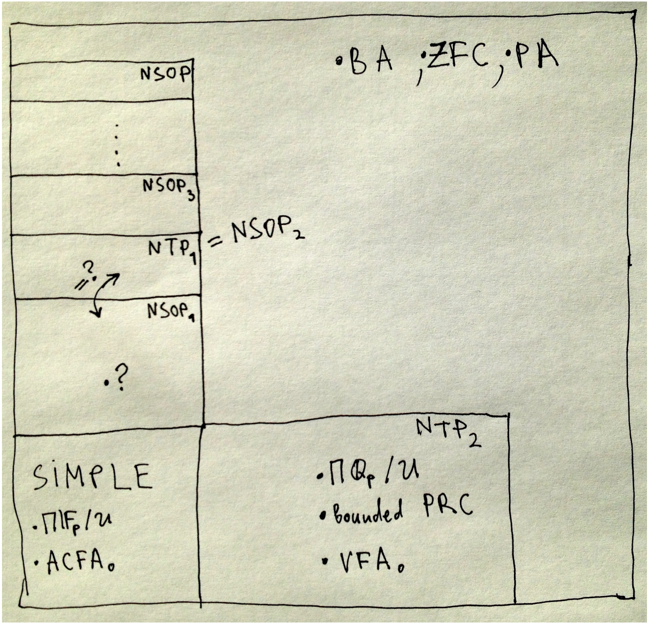
Artem Chernikov
(IMJ-PRG)

Paris,

June 9, 2015

Joint work with Nicholas Ramsey (UC Berkeley).

Shelah's classification



Tree properties

Let T be a complete theory and $\varphi(x; y) \in L$ a formula in the language of T.

- $\varphi(x; y)$ has the *tree property* (TP) if there is $k < \omega$ and a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in $\mathbb{M}$ such that:
  - for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
  - for all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta \frown \langle i \rangle}) : i < \omega\}$ is k-inconsistent.
- $\varphi(x; y)$ has the *tree property of the first kind* (TP_1) if there is a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in $\mathbb{M}$ such that:
  - for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
  - for all $\eta \perp \nu$ in $\omega^{<\omega}$, $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
- $\varphi(x; y)$ has the *tree property of the second kind* (TP_2) if there is a $k < \omega$ and an array $(a_{\alpha,i})_{\alpha<\omega, i<\omega}$ in $\mathbb{M}$ such that:
  - for all functions $f : \omega \to \omega$, $\{\varphi(x; a_{\alpha,f(\alpha)}) : \alpha < \omega\}$ is consistent,
  - for all α, $\{\varphi(x; a_{\alpha,i}) : i < \omega\}$ is k-inconsistent.
- T has one of the above properties if some formula does modulo T.

Shelah's theorem, 1

- So TP_1 and TP_2 are two extreme forms in which TP can occur. In TP_1, everything that is not forced to be consistent by the definition of TP, is inconsistent. In TP_2, everything that is not forced to be inconsistent by the definition of TP, is consistent.

Fact

[Shelah] If T has TP*, then it either has* TP_1 *or* TP_2*.*

- To each theory T, one associates cardinal invariants $\kappa_{\mathrm{cdt}}, \kappa_{\mathrm{sct}}, \kappa_{\mathrm{inp}}$ measuring how much of TP, TP_1 and TP_2 (respectively) it contains. Namely, we allow different formulas at each level in the definition above, and take the first cardinal such that there is no tree with that many levels.
- E.g. $\kappa_{\mathrm{cdt}} = \infty$ iff T has TP, and T is supersimple iff $\kappa_{\mathrm{cdt}} = \aleph_0$. Similarly, $\kappa_{\mathrm{inp}} = \infty$ iff T has TP_2, and T is strong iff $\kappa_{\mathrm{inp}} = \aleph_0$.
- Shelah asked for a quantitative refinement of the above theorem: does $\kappa_{\mathrm{cdt}} = \kappa_{\mathrm{sct}} + \kappa_{\mathrm{inp}}$ hold?

Shelah's theorem, 2

Theorem

If T is countable, then $\kappa_{\mathrm{cdt}} = \kappa_{\mathrm{sct}} + \kappa_{\mathrm{inp}}$.

- In fact if T is countable, then $\kappa_{\mathrm{cdt}}, \kappa_{\mathrm{sct}}, \kappa_{\mathrm{inp}} \in \{\aleph_0, \aleph_1, \infty\}$. We treat each of $\aleph_0$ and $\aleph_1$ separately, the ∞ case follows from Shelah's theorem.

Theorem

[Ramsey] There are theories (in an uncountable language) with $\kappa_{\mathrm{cdt}} > \kappa_{\mathrm{inp}} + \kappa_{\mathrm{sct}}$.

- Constructs a theory reducing the question to a deep result of Shelah and Juhász on the non-existence of homogeneous partitions for certain colorings of families of finite subsets of certain cardinals (one can take $\kappa = \left(2^{\lambda}\right)^{++} + \omega_4$ for some infinite cardinal λ, then there is T with $|T| = \kappa$ and such that $\kappa_{\mathrm{cdt}} = \kappa^{+}$ but $\kappa_{\mathrm{sct}} \leq \kappa$ and $\kappa_{\mathrm{inp}} \leq \kappa$).

So what is known about NTP_1?

- [Kim, Kim] In the definition of TP_1, one can replace 2-inconsistency by k-inconsistency, for any $k \geq 2$. Also, there is a characterization of NTP_1 via counting certain families of partial types.
- [Malliaris, Shelah] If T has TP_1, then it is maximal in the Keisler order (via equivalence to SOP_2, see later).
- Not much more. For example, any kind of a basic theory of forking is missing.
- Another question from Shelah's book, in the special case: is TP_1 always witnessed by a formula in a single variable?
- As usual for this kind of questions, to simplify combinatorics we would like to work with "indiscernible" witnesses of our properties.

Indiscernible trees, 1

- Fix a theory T in a language L and $\mathbb{M} \models T$ a monster model.
- Consider the language $L_0 = \{\triangleleft, \wedge, <_{lex}\}$. We view the tree $\kappa^{<\lambda}$ as an L_0-structure in a natural way, interpreting $\triangleleft$ as the tree partial order, $\wedge$ as the binary meet function and $<_{lex}$ as the lexicographic order.
- Suppose that $(a_\eta)_{\eta \in \kappa^{<\lambda}}$ is collection of tuples and C a set of parameters in some model.
- We say that $(a_\eta)_{\eta \in \kappa^{<\lambda}}$ is a *strongly indiscernible tree over* C if

$$\mathrm{qftp}_{L_0}(\eta_0, \dots, \eta_{n-1}) = \mathrm{qftp}_{L_0}(\nu_0, \dots, \nu_{n-1})$$

implies $\mathrm{tp}_L(a_{\eta_0}, \dots, a_{\eta_{n-1}}/C) = \mathrm{tp}_L(a_{\nu_0}, \dots, a_{\nu_{n-1}}/C)$, for all $n \in \omega$.

Indiscernible trees, 2

Using some results from structural Ramsey theory of trees, one can show that indiscernible trees "exist". More precisely, let I_0 be the L_0-structure $(\omega^{<\omega}, \trianglelefteq, <_{lex}, \wedge)$ with all symbols given their intended interpretations.

Fact

[Takeuchi, Tsuboi], [Kim, Kim, Scow] Given any tree $(a_i : i \in I_0)$ of tuples from $\mathbb{M}$, there is a strongly indiscernible tree $(b_i : i \in I_0)$ in $\mathbb{M}$ locally based on the (a_i): given any finite set of formulas Δ from L and a finite tuple $(t_0, \ldots, t_{n-1})$ from I_0, there is a tuple $(s_0, \ldots, s_{n-1})$ from I_0 such that

$$\mathrm{qftp}_{L_0}(t_0, \ldots, t_{n-1}) = \mathrm{qftp}_{L_0}(s_0, \ldots, s_{n-1})$$

and

$$tp_\Delta(b_{t_0}, \ldots, b_{t_{n-1}}) = tp_\Delta(a_{s_0}, \ldots, a_{s_{n-1}}).$$

Path collapse lemma, 1

- In particular, if $\phi(x; y)$ has TP_1, then there is a strongly indiscernible tree witnessing this.
- (Path Collapse lemma) Suppose κ is an infinite cardinal, $(a_\eta)_{\eta \in 2^{<\kappa}}$ is a tree strongly indiscernible over a set of parameters C and, moreover, $(a_{0^\alpha} : 0 < \alpha < \kappa)$ is an indiscernible sequence over cC. Let

$$p(y; \bar{z}) = \mathrm{tp}(c; (a_{0 \frown 0^\gamma} : \gamma < \kappa)/C).$$

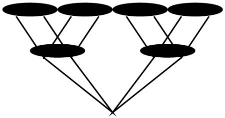
Then if

$$p(y; (a_{0 \frown 0^\gamma})_{\gamma<\kappa}) \cup p(y; (a_{1 \frown 0^\gamma})_{\gamma<\kappa})$$

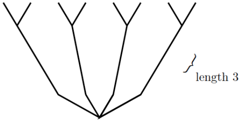
is not consistent, then T has TP_1, witnessed by a formula with free variables y.

Path collapse lemma, 2

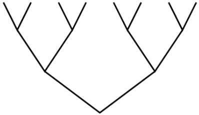
The proof requires in particular a (rather tedious) demonstration that various operations on strongly indiscernible trees preserve strong indiscernibility, e.g.



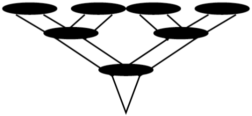
2-fold widening at level 1



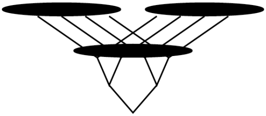
3-fold stretch at level 1



0-fold fattening



1-fold fattening



2-fold fattening

Application 1: TP_1 is witnessed by a formula in a single variable

Theorem

Suppose T witnesses TP_1 via $\varphi(x, y; z)$. Then there is a formula $\varphi_0(x; v)$ with free variables x and parameter variables v, or a formula $\varphi_1(y; w)$ with free variables y and parameter variables w so that one of φ_0 and φ_1 witness TP_1.

- Proof idea. Start with a strongly indiscernible tree witnessing that φ has TP_1. Assume that no formula in the free variable y has TP_1, and let bc_0 realize a branch of the tree. Then iteratively applying the path collapse lemma to the type of c_0 over that branch in increasing fattenings of the tree we can conclude by compactness that there is some c such that $\varphi(x; c, z)$ has TP_1, which is enough.

Application 2: Weak $k - \mathrm{TP}_1$ is equivalent to TP_1

- Say that a subset $\{\eta_i : i < k\} \subseteq \omega^{<\omega}$ is a collection of *distant siblings* if given $i \neq i'$, $j \neq j'$, all of which are $< k$, $\eta_i \wedge \eta_{i'} = \eta_j \wedge \eta_{j'}$.

Definition

[Kim, Kim] $\varphi(x; y)$ has *weak* $k - \mathrm{TP}_1$ if there is a a collection of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that:

- for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent.
- if $\{\eta_i : i < k\} \subseteq \omega^{<\omega}$ is a collection of distinct distant siblings, then $\{\varphi(x; a_{\eta_i}) : i < k\}$ is inconsistent.

- $\mathrm{TP}_1 \iff$ weak 2-$\mathrm{TP}_1 \implies$ weak 3-$\mathrm{TP}_1 \implies \dots$
- [Kim, Kim] Do the converse implications hold?

Theorem

T has weak k-TP_1 iff it has TP_1, for all $k \geq 2$.

SOP_n hierarchy, 1

Definition

[Shelah], [Dzamonja, Shelah]

- Fix $n \geq 3$. We say that a formula $\phi(x; y)$ has SOP_n if:
  - there are pairwise different $(a_i)_{i \in \omega}$ such that $\models \phi(a_i, a_j)$ for all $i < j < \omega$,
  - $\models \neg \exists x_0 \ldots x_{n-1} \bigwedge_{j=i+1 (\mod n)} \phi(x_i, x_j)$.
- $\varphi(x; y)$ has SOP_2 if there is a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ such that:
  - for all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
  - If $\eta, \nu \in 2^{<\omega}$ and $\eta \perp \nu$, then $\{\varphi(x; a_\eta), \varphi(x; a_\nu)\}$ is inconsistent.
- $\varphi(x; y)$ has SOP_1 if there are $(a_\eta)_{\eta \in 2^{<\omega}}$ such that:
  - for all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|n}) : n < \omega\}$ is consistent,
  - if $\eta \frown 0 \trianglelefteq \nu \in 2^{<\omega}$, then $\{\varphi(x; a_{\eta \frown 1}), \varphi(x; a_\nu)\}$ is inconsistent.
- Motivated by the Keisler order and related questions.

SOP_n hierarchy, 2

- What is known:
  - $\mathrm{NTP} \subseteq \mathrm{NSOP}_1 \subseteq \mathrm{NSOP}_2 = \mathrm{NTP}_1 \subseteq \mathrm{NSOP}_3 \subseteq \ldots \subseteq \mathrm{NSOP}$.
  - $\mathrm{NSOP}_{n+1} \setminus \mathrm{NSOP}_n \neq \emptyset$ for all $n \geq 3$, and $\mathrm{NSOP} \setminus (\bigcup_n \mathrm{NSOP}_n) \neq \emptyset$.
  - $\mathrm{NSOP}_2 \cap \mathrm{NTP}_2 = \mathrm{NTP}$ (Shelah's theorem).
  - [Shelah, Usvyatsov] give an example showing that $\mathrm{NTP} \subsetneq \mathrm{NSOP}_1$, however their proof appears to be wrong. Yet their example is correct, as follows from our theorem.
- Open problems:
  - $\mathrm{NSOP}_2 \subsetneq \mathrm{NSOP}_3$? $\mathrm{NSOP}_1 \subsetneq \mathrm{NSOP}_2$?
  - Does $\mathrm{NSOP}_n \cap \mathrm{NTP}_2$ collapse for $n \geq 3$? At least, $\mathrm{NTP} \subsetneq \mathrm{NSOP} \cap \mathrm{NTP}_2$?

Independent amalgamation of types

- Suppose $\mathop{\smile\!\!\!\!\!\!|}$ is an Aut($\mathbb{M}$)-invariant ternary relation on small subsets of $\mathbb{M}$.

Definition

1. $\mathop{\smile\!\!\!\!\!\!|}$ satisfies *weak independent amalgamation* over models if, given $M \models T$, $b_0c_0 \equiv_M b_1c_1$ satisfying $b_i \mathop{\smile\!\!\!\!\!\!|}_M c_i$ for $i = 0, 1$ and $c_0 \mathop{\smile\!\!\!\!\!\!|}_M c_1$, there is b satisfying $bc_0 \equiv_M bc_1 \equiv_M b_0c_0$.
2. $\mathop{\smile\!\!\!\!\!\!|}$ satisfies *independent amalgamation* over models if, given $M \models T$, $b_0 \equiv_M b_1$ satisfying $b_i \mathop{\smile\!\!\!\!\!\!|}_M c_i$ for $i = 0, 1$ and $c_0 \mathop{\smile\!\!\!\!\!\!|}_M c_1$, there is b satisfying $bc_0 \equiv_M b_0c_0$ and $bc_1 \equiv_M b_1c_1$.
3. $\mathop{\smile\!\!\!\!\!\!|}$ satisfies *stationarity* over models if, given $M \models T$, if $b_0 \equiv_M b_1$ and $b_0 \mathop{\smile\!\!\!\!\!\!|}_M c, b_1 \mathop{\smile\!\!\!\!\!\!|}_M c$ then $b_0 \equiv_{Mc} b_1$.

- Stationarity $\implies$ independent amalgamation $\implies$ weak independent amalgamation.
- E.g. $\mathop{\smile\!\!\!\!\!\!|}^f$ satisfies stationarity over models in stable theories and independent amalgamation in simple theories.

Weak independent amalgamation and NSOP_1

Suppose A, B, C are small subsets of the monster $\mathbb{M}$.

- $A \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{C} B$ if and only if $\mathrm{tp}(A/BC)$ can be extended to a global type invariant over C. We denote its dual by $\mathop{\smile\!\!\!\!\!\!\mid}^{ci}$ — i.e. $A \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{C} B$ holds if and only if $B \mathop{\smile\!\!\!\!\!\!\mid}^{ci}_{C} A$.
- $A \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{C} B$ if and only if $\mathrm{tp}(A/BC)$ is finitely satisfiable in C. We denote its dual by $\mathop{\smile\!\!\!\!\!\!\mid}^{h}$ — i.e. $A \mathop{\smile\!\!\!\!\!\!\mid}^{h}_{C} B$ if and only if $B \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{C} A$.

Theorem

The following are equivalent.

1. *T is* NSOP_1.
2. $\mathop{\smile\!\!\!\!\!\!\mid}^{ci}$ *satisfies weak independent amalgamation: given any $M \models T$, $b_0c_0 \equiv_M b_1c_1$ so that $c_1 \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{M} c_0$ and $c_j \mathop{\smile\!\!\!\!\!\!\mid}^{i}_{M} b_j$ for $j = 0, 1$, there is b so that $bc_0 \equiv_M bc_1 \equiv_M b_0c_0$.*
3. $\mathop{\smile\!\!\!\!\!\!\mid}^{h}$ *satisfies weak independent amalgamation: given any $M \models T$, $b_0c_0 \equiv_M b_1c_1$ so that $c_1 \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} c_0$ and $c_j \mathop{\smile\!\!\!\!\!\!\mid}^{u}_{M} b_j$ for $j = 0, 1$, there is b so that $bc_0 \equiv_M bc_1 \equiv_M b_0c_0$.*

A sufficient criterion for $NSOP_1$

Corollary

Assume there is an $Aut(\mathbb{M})$-invariant independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on small subsets of the monster $\mathbb{M} \models T$ such that it satisfies the following properties, for an arbitrary $M \models T$.

1. *Strong finite character: if $a \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$, then there is a formula $\varphi(x, b, m) \in tp(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \not\mathop{\smile\!\!\!\!\!\!\mid}_M b$.*
2. *Existence over models: $M \models T$ implies $a \mathop{\smile\!\!\!\!\!\!\mid}_M M$ for any a.*
3. *Monotonicity: $aa' \mathop{\smile\!\!\!\!\!\!\mid}_M bb' \implies a \mathop{\smile\!\!\!\!\!\!\mid}_M b$.*
4. *Symmetry: $a \mathop{\smile\!\!\!\!\!\!\mid}_M b \iff b \mathop{\smile\!\!\!\!\!\!\mid}_M a$.*
5. *Independent amalgamation: $c_0 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1, b_0 \mathop{\smile\!\!\!\!\!\!\mid}_M c_0, b_1 \mathop{\smile\!\!\!\!\!\!\mid}_M c_1, b_0 \equiv_M b_1$ implies there exists b with $b \equiv_{c_0 M} b_0, b \equiv_{c_1 M} b_1$.*

Then T is $NSOP_1$.

- We **do not** require local character, and strong finite character **cannot** be relaxed to finite character.

Examples of NSOP$_1$ theories: vector spaces with a generic bilinear form, 1

- Let L denote the language with two sorts V and K containing the language of abelian groups for variables from V, the language of rings for variables from K, a function $\cdot : K \times V \to V$, and a function $[\,] : V \times V \to K$.
- T_∞ is the model companion of the L-theory asserting that K is a field, V is a K-vector space of infinite dimension with the action of K given by $\cdot$, and $[\,]$ is a non-degenerate bilinear form on V.
- If $(K, V) \models T_\infty$ then K is an algebraically closed field.

The theory T_∞ was introduced by Nicolas Granger, who observed that its completions are not simple, but nonetheless have a notion of independence called Γ-non-forking satisfying essentially all properties of forking in stable theories, except local character.

Examples of $NSOP_1$ theories: vector spaces with a generic bilinear form, 2

Let $M = (V, \tilde{K})$ be a sufficiently saturated model of T_∞. Let $A \subseteq B \subset M$ and $c \in M$ with c a singleton. Let $c \mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}_{A} B$ be the assertion that $K_{Ac} \mathop{\smile\!\!\!\!\!\!\mid}^{ACF}_{K_A} K_B$ in the sense of non-forking independence for algebraically closed fields and one of the following holds: $c \in \tilde{K}$; $c \in \langle A \rangle$; $c \notin \langle B \rangle$and $[c, B]$ is Φ-independent over A, where "$[c, B]$ is Φ-independent over A" means that whenever $\{b_0, \ldots, b_{n-1}\}$ is a linearly independent set in $B_V \cap (V \setminus \langle A \rangle)$ then the set $\{[c, b_0], \ldots, [c, b_{n-1}]\}$ is algebraically independent over the field $K_B(K_{Ac})$.
By induction, for $c = (c_0, \ldots, c_m)$ define $c \mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}_{A} B$ by

$$c \mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}_{A} B \iff (c_0, \ldots, c_{m-1}) \mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}_{A} B \text{ and } c_m \mathop{\smile\!\!\!\!\!\!\mid}^{\Gamma}_{Ac_0 \ldots c_{m-1}} Bc_0 \ldots c_{m-1}.$$

Examples of $NSOP_1$ theories: vector spaces with a generic bilinear form, 3

- [Granger] Let $M = (V, K) \models T_\infty$. Then the relation on subsets of M given by Γ-non-forking is automorphism invariant, symmetric, and transitive. Moreover, it satisfies extension, finite character, and stationarity over a model.
- Moreover, it is not hard to check that Γ-non-forking satisfies strong finite character.
- Applying the criterion, we conclude that T_∞ is $NSOP_1$.

Examples of NSOP_1 theories: ω-free PAC fields of char 0

- A field F is *pseudo-algebraically closed* (or *PAC*) if every absolutely irreducible variety defined over F has an F-rational point. A field F is called ω*-free* if it has a countable elementary substructure F_0 with $\mathcal{G}(F_0) \cong \hat{\mathbb{F}}_\omega$, the free profinite group on countably many generators.
- [Chatzidakis] A PAC field has a simple theory if and only if it has finitely many degree n extensions for all n, so an ω-free PAC field is not simple.
- [Chatzidakis] Suppose F is a sufficiently saturated ω-free PAC field of characteristic 0. Given $A = \mathrm{acl}(A)$, $B = \mathrm{acl}(B)$, $C = \mathrm{acl}(C)$ with $C \subseteq A, B \subseteq F$, write $A \mathop{\smile\!\!\!\!\!\mid}^I_C B$ to indicate that $A \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ACF}}_C B$ and $A^{\mathrm{alg}} B^{\mathrm{alg}} \cap \mathrm{acl}(AB) = AB$. Extend this to non-algebraically closed sets by stipulating $a \mathop{\smile\!\!\!\!\!\mid}^I_D b$ holds if and only if $\mathrm{acl}(aD) \mathop{\smile\!\!\!\!\!\mid}^I_{\mathrm{acl}(D)} \mathrm{acl}(bD)$. Then $\mathop{\smile\!\!\!\!\!\mid}^I$ satisfies existence over models, monotonicity, symmetry, and independent amalgamation over models. Strong finite character holds as well. It follows that F is NSOP_1.

References

- S. Shelah. "Classification theory and the number of nonisomorphic models", volume 92 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, second edition, 1990.
- Saharon Shelah, Alex Usvyatsov. "More on SOP1 and SOP2". Annals of Pure and Applied Logic, 155:16–31, 2008.
- Byunghan Kim, Hyeung-Joon Kim. "Notions around tree property 1". Ann. Pure Appl. Logic, 162(9):698–709, 2011.
- Kota Takeuchi, Akito Tsuboi. "On the existence of indiscernible trees". Ann. Pure Appl. Logic, 163(12):1891–1902, 2012.
- Artem Chernikov, Nicholas Ramsey, "On model-theoretic tree properties", arXiv:1505.00454.
- Nicholas Ramsey, "A counterexample to $\kappa_{\mathsf{cdt}} = \kappa_{\mathsf{sct}} + \kappa_{\mathsf{inp}}$", preprint.