Recognizing groups and fields in Erdős geometry and model theory

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Some practical and structural work in Oaxaca.

- The *trichotomy principle* in model theory: in a sufficiently tame context (certain strongly minimal, *o*-minimal), every structure is either "trivial", or essentially a vector space ("modular"), or interprets a field.
- Asymptotic sizes of the intersections of definable sets with finite grids in certain model-theoretically tame contexts reflect this trichotomy principle, and detect presence of algebraic structures (groups, fields).
- Instances of this principle are well-known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures. Here we discuss "inverse" theorems which show this is the only way.

Sum-product and expander polynomials

- [Erdős, Szemerédi'83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,

$$\max\{|A+A|, |A \cdot A|\} = \Omega\left(|A|^{1+c}\right).$$

- [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any ε).
- [Elekes, Rónyai'00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree d, then for all $A, B \subseteq_n \mathbb{R}$,

$$|f(A \times B)| = \Omega_d\left(n^{\frac{4}{3}}\right),$$

unless f is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials g, h, i.

Elekes-Szabó theorem

- [Elekes-Szabó'12] provide a conceptual generalization: for any algebraic surface $R(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:
    1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq_n \mathbb{R}$ we have
    $$|R \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}).$$
    2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \to V$ such that
    $$\pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \Leftrightarrow R(x_1, x_2, x_3)$$
    for all $x_i \in U_i$.

Generalizations of the Elekes-Szabó theorem

Let $R \subseteq X_1 \times \ldots \times X_r$ be an algebraic surface (or just a definable set) with finite-to-one projection onto any $r-1$ coordinates and $\dim(X_i) = m$.

1. [Elekes, Szabó'12] $r = 3$, m arbitrary over $\mathbb{C}$ (only count on grids in *general position*, correspondence with a complex algebraic group of dimension m);
2. [Raz, Sharir, de Zeeuw'18] $r = 4$, $m = 1$ over $\mathbb{C}$;
3. [Raz, Shem-Tov'18] $m = 1$, R of the form $f(x_1, ..., x_{r-1}) = x_r$ for any r over $\mathbb{C}$.
4. [Hrushovski'13] Pseudofinite dimension, modularity
5. [Bays, Breuillard'18] r and m arbitrary over $\mathbb{C}$, recognized that the arising groups are abelian (no bounds on γ);
6. Further work: [Wang'15]; [Bukh, Tsimmerman' 12], [Tao'12]; [Jing, Roy, Tran'19], [Bays, Dobrowolski, Zou].
7. [C., Peterzil, Starchenko] Any r and m, any o-minimal structure or stable with a distal expansion and explicit bounds on γ. A special case:

One-dimensional o-minimal case

Theorem (C., Peterzil, Starchenko)

Assume $r \geq 3$, $\mathcal{M}$ is an o-minimal expansion of $\mathbb{R}$ and $R \subseteq \mathbb{R}^r$ is definable, such that the projection of R to any $r-1$ coordinates is finite-to-one. Then exactly one of the following holds.

1. *For any finite $A_i \subseteq_n \mathbb{R}$, $i \in [r]$, we have*

$$|R \cap (A_1 \times \ldots \times A_r)| = O_R\left(n^{r-1-\gamma}\right),$$

   where $\gamma = \frac{1}{3}$ if $r \geq 4$, and $\gamma = \frac{1}{6}$ if $r = 3$.

2. *There exist open sets $U_i \subseteq \mathbb{R}, i \in [r]$, an open set $V \subseteq \mathbb{R}$ containing 0, and homeomorphisms $\pi_i : U_i \to V$ such that*

$$\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \Leftrightarrow R(x_1, \ldots, x_r)$$

   for all $x_i \in U_i, i \in [r]$.

General o-minimal case

Theorem (C., Peterzil, Starchenko)

*Let $\mathcal{M}$ be an o-minimal expansion of $\mathbb{R}$. Assume $r \geq 3$, $R \subseteq X_1 \times \cdots \times X_r$ are definable with $\mathbf{dim}\,(\boldsymbol{X_i}) = \boldsymbol{m}$, and the projection of R to any $r-1$ coordinates is finite-to-one. Then ~~**exactly**~~ one of the following holds.*

1. *For any finite $A_i \subseteq_n X_i$ **in general position**, $i \in [r]$, we have*

$$|R \cap (A_1 \times \ldots \times A_r)| = O_R\left(n^{r-1-\gamma}\right),$$

   for $\gamma = \frac{1}{8m-5}$ if $r \geq 4$, and $\gamma = \frac{1}{16m-10}$ if $r = 3$.

2. *There exist definable relatively open sets $U_i \subseteq X_i$, $i \in [r]$, **an abelian Lie group** $(G, +)$ **of dimension** m and an open neighborhood $V \subseteq G$ of 0, and definable homeomorphisms $\pi_i : U_i \to V$, $i \in [r]$, such that for all $x_i \in U_i, i \in [r]$*

$$\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \Leftrightarrow R(x_1, \ldots, x_r).$$

Remarks

1. If $\mathcal{M}$ is *o*-minimal but is not elementarily equivalent to an expansion of $\mathbb{R}$ — only get correspondence with a type-definable group.
2. One ingredient — "Sźemeredi-Trotter"-style bounds in *o*-minimal, and more generally *distal* structures.
3. Another – a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a "generic chunk", along with a purely combinatorial version (providing a "coordinatization" result for the associated locally modular pregeometry).

First ingredient: Recognizing groups, 1

1. Assume that $(G, +, 0)$ is an abelian group, and consider the r-ary relation $R \subseteq \prod_{i \in [r]} G$ given by $x_1 + \ldots + x_r = 0$.
2. Then R is easily seen to satisfy the following two properties, for any permutation of the variables of R:

$$\forall x_1, \ldots, \forall x_{r-1} \exists! x_r R(x_1, \ldots, x_r), \tag{P1}$$

$$\forall x_1, x_2 \forall y_3, \ldots y_r \forall y_3', \ldots, y_r' \Big(R(\bar{x}, \bar{y}) \wedge R(\bar{x}, \bar{y}') \rightarrow \tag{P2}$$

$$\big(\forall x_1', x_2' R(\bar{x}', \bar{y}) \leftrightarrow R(\bar{x}', \bar{y}') \big) \Big).$$

We show a converse, assuming $r \geq 4$:

Recognizing groups, 2

Theorem (C., Peterzil, Starchenko)

Assume $r \in \mathbb{N}_{\geq 4}$, $X_1, \ldots, X_r$ and $R \subseteq \prod_{i \in [r]} X_i$ are sets, so that R satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group $(G, +, 0_G)$ and bijections $\pi_i : X_i \to G$ such that for every $(a_1, \ldots, a_r) \in \prod_{i \in [r]} X_i$ we have

$$R(a_1, \ldots, a_r) \iff \pi_1(a_1) + \ldots + \pi_r(a_r) = 0_G.$$

- If $X_1 = \ldots = X_r$, property (P1) is equivalent to saying that the relation R is an $(r-1)$-*dimensional permutation* on the set X_1, or a *Latin* $(r-1)$-*hypercube*, as studied by Linial and Luria. Thus the condition (P2) characterizes, for $r \geq 3$, those Latin r-hypercubes that are given by the relation "$x_1 + \ldots + x_{r-1} = x_r$" in an abelian group.
- If R is definable and X_i are type-definable in a (saturated) $\mathcal{M}$, then G is type-definable and π_i are relatively definable in $\mathcal{M}$.

Recognizing groups in the stable case

- In the stable version of our theorem, we only get "generic correspondence" with a type-definable group.
- An *r-gon* is a tuple $a_1, \ldots, a_r$ such that any $r-1$ of its elements are (forking-)independent, and any element in it is in the algebraic closure of the other ones.
- An r-gon is *abelian* if, after any permutation of its elements, we have $a_1 a_2 \mathop{\smile\!\!\!\!\!\!\perp}_{\operatorname{acl}(a_1 a_2) \cap \operatorname{acl}(a_3 \ldots a_r)} a_3 \ldots a_r$.
- If $(G, \cdot)$ is a type-definable abelian group, $g_1, \ldots, g_{r-1}$ are independent generics in G and $g_r := g_1 \cdot \ldots \cdot g_{r-1}$, then $g_1, \ldots, g_r$ is an abelian r-gon (associated to G).
- Conversely,

Theorem (C., Peterzil, Starchenko; independently Hrushovski)

Let $r \geq 4$ and $a_1, \ldots, a_r$ be an abelian r-gon. Then there is a type-definable (in $\mathcal{M}^{\mathrm{eq}}$) connected abelian group $(G, \cdot)$ and an abelian r-gon $g_1, \ldots, g_s$ associated to G, such that after a base change each g_i is interalgebraic with a_i.

Second ingredient: distality

Definition

A structure $\mathcal{M}$ is *distal* if and only if for every definable family $\{\varphi(x,b) : b \in M_y\}$ of subsets of M_x there is a definable family $\left\{\theta(x,c) : c \in M_y^k\right\}$ such that for every $a \in M_x$ and every finite set $B \subset M_y$ there is some $c \in B^k$ such that:

- $a \models \theta(x,c)$;
- $\theta(x,c) \vdash \mathrm{tp}_\varphi(a/B)$, that is for every $a' \models \theta(x,c)$ and $b \in B$ we have $a' \models \phi(x,b) \Leftrightarrow a \models \phi(x,b)$.

Examples of distal structures

- $\mathcal{M}$ distal $\implies$ $\mathcal{M}$ is NIP, unstable.
- Examples of distal structures: (weakly) *o*-minimal structures, various valued fields of char 0 (e.g. $\mathbb{Q}_p$, RCVF, the valued differential field of transseries).
- Stable structures with distal expansions: ACF_0, $\mathrm{DCF}_{0,m}$, CCM, abelian groups, Hrushovski constructions*.
- Stable structures without distal expansions: ACF_p [C., Starchenko'15], a disjoint union of finite expander graphs (e.g. Ramanujan graphs) of growing degree and expansion [Jiang, Nesetril, Ossona de Mendez, Siebertz'20], any Banach space [Hanson] (distality in continuous logic is developed by [Anderson]).
- **Problem.** Do non-abelian free groups have distal expansions?

Number of edges in a $K_{k,\dots,k}$-free hypergraph

- The following fact is due to [Kővári, Sós, Turán'54] for $r = 2$ and [Erdős'64] for general r.

Fact (The Basic Bound)

If H is a $K_{k,\dots,k}$-free r-hypergraph then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$.

- So the exponent is slightly better than the maximal possible r (we have n^r edges in $K_{n,\dots,n}$). A probabilistic construction in [Erdős'64] shows that it cannot be substantially improved.

Bounds for graphs definable in distal structures

- Generalizing [Fox, Pach, Sheffer, Suk, Zahl'15] in the semialgebraic case, we have:

Fact (C., Galvin, Starchenko'16)

Let $\mathcal{M}$ be a distal structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists some $\varepsilon = \varepsilon(R, k) > 0$ such that for any $A_1 \subseteq_n M_{x_1}, A_2 \subseteq_n M_{x_2}$, if $E := R \cap (A_1 \times A_2)$ is $K_{k,k}$-free then $|E| = O_{R,k}(n^{t-\varepsilon})$, where t is the exponent given by the Basic Bound for arbitrary graphs.

- In fact, ε is given in terms of k and the size of the smallest *distal cell decomposition* for R.
- E.g. if $R \subseteq M^2 \times M^2$ for an *o*-minimal $\mathcal{M}$, then $t - \varepsilon = \frac{4}{3}$ ([C., Galvin, Starchenko'16]; independently, [Basu, Raz'16]).

Recognizing fields

- For the semialgebraic $K_{2,2}$-free point-line incidence relation $R = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1 x_1 + y_2\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have the (optimal) lower bound $|R \cap (V_1 \times V_2)| = \Omega(n^{\frac{4}{3}})$.
- To define it we use both addition and multiplication, i.e. the field structure.
- This is not a coincidence — any non-trivial lower bound on the Zarankiewicz exponent of R allows to recover a field from it:

Theorem (Basit, C., Starchenko, Tao, Tran)

Assume that $\mathcal{M} = (M, <, \ldots)$ is o-minimal and $R \subseteq M_{x_1} \times \ldots \times M_{x_r}$ is a definable relation which is $K_{k,\ldots,k}$-free, but $|R \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})$ for $V_i \subseteq_n M_{x_i}$. Then a real closed field is definable in the first-order structure $(M, <, R)$.

Ingredients

- An (almost) optimal bound on the number of edges in $K_{k,\dots,k}$-free hypergraphs definable in locally modular *o*-minimal expansions of groups, so e.g. for semilinear ($=$ definable in $(\mathbb{R}, <, +)$) hypergraphs.
- The trichotomy theorem for *o*-minimal structures [Peterzil, Starchenko'98].

Local modularity

- We write $a \mathop{\smile\!\!\!\!\!\!|}_C B$ to denote that $\dim(a/BC) = \dim(a/C)$ in the matroid associated to the algebraic closure in an *o*-minimal structure.
- An *o*-minimal structure is *(weakly) locally modular* if for any small subsets $A, B \subseteq \mathbb{M} \models T$ there exists some small set $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(AC) \cap \mathrm{acl}(BC)} B$.
- Intuition: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.
- In particular, an *o*-minimal structure is locally modular if and only if any normal interpretable family of plane curves in T has dimension ≤ 1.

Bound for semilinear relations

Theorem (Basit, C., Starchenko, Tao, Tran)

Let $\mathcal{M}$ be an o-minimal locally modular expansion of a group and Q a definable relation of arity $r \geq 2$. Then for any $\varepsilon > 0$ and any V_i with $|V_i| = n$ such that $E := Q \cap V_1 \times \ldots \times V_r$ is $K_{k,\ldots,k}$-free, we have

$$|E| = O_{Q,k,\varepsilon}\left(n^{r-1+\varepsilon}\right).$$

Moreover, if Q itself is $K_{k,\ldots,k}$-free, then for any V_i with $|V_i| = n$ we have

$$|E| = O_Q(n^{r-1}).$$

Recovering a field in the *o*-minimal case

Fact (Peterzil, Starchenko'98)

Let $\mathcal{M}$ be an o-minimal (saturated) structure. TFAE:

- *$\mathcal{M}$ is not locally modular;*
- *there exists a real closed field definable in $\mathcal{M}$.*

- [Marker, Peterzil, Pillay'92] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\cdot \restriction_{[0,1]^2}$ is definable in $(\mathbb{R}, <, +, X)$. In particular, it is not locally modular.
- Combining this with the "optimal" bound in the locally modular case, we get the result.
- Problem: is it possible to establish a more direct correspondence between the relation with many edges and the point-line incidence relation in a field?

An application to incidences with polytopes

- Applying with $r = 2$ we get the following:

Corollary

For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.
Let $d \in \mathbb{N}$ and $H_1, \ldots, H_s \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_s$.
For any set V_1 of n_1 points in $\mathbb{R}^d$ and any set V_2 of n_2 polytopes in $\mathcal{F}$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $\alpha n (\log n)^s$ incidences.

- In particular (this corollary was obtained independently by [Tomon, Zakharov]):

Corollary

For any set V_1 of n_1 points and any set V_2 of n_2 (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k}\left(n(\log n)^{2d}\right)$ incidences.

Dyadic rectangles and a lower bound

- Is the logarithmic factor necessary?
- Using a different argument, restricting to dyadic boxes on the plane ($d = 2$), we gave a stronger upper bound $O\left(n\frac{\log n}{\log\log n}\right)$, and give a construction showing a matching lower bound (up to a constant).
- [Tomon, Zakharov] use our construction to disprove a conjecture of Alon, Basavaraju, Chandran, Mathew, and Rajendraprasad regarding the maximal possible number of edges in a graph of bounded separation dimension.
- We asked what happens for higher d?
- Very recently, [Har-Peled, Chan] improve our upper bound to $O\left(n\left(\frac{\log n}{\log\log n}\right)^{d-1}\right)$, and point out that a result of [Chazelle'90] on "data structures" shows its optimality for all d!

Congrats Kobi and Sergei!