Higher classification theory and n-amalgamation

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N-Tameness, 1

1. Tameness notions in Shelah's classification are typically given by restrictions on the combinatorial complexity of definable binary relations, by forbidding certain induced subgraphs (e.g. T is *stable* if no definable binary relation can contain arbitrary large finite half-graphs; and *NIP* if sufficiently large random bipartite graphs are omitted; and distal if bipartite "expanders" are omitted).
2. A typical result then demonstrates that binary relations are "approximated" by the unary ones, up to a "small" error. For example, stationarity of forking in stable theories says that given $p(x), q(y)$ types over a model M, there exists a *unique type* $r(x, y)$ over M so that if $(a, b) \models r$ then $a \models p, b \models q$ and $a \mathop{\smile\!\!\!\!\!\!|}_M b$ — that is, there is a unique type $r(x, y)$ extending $p(x) \cup q(y)$, up to the forking formulas $\varphi(x, y) \in \mathcal{L}(M)$.

N-tameness, 2

1. Another example: T is distal if and only if for any $p(x), q(y)$ global invariant types that commute, there is a unique global type $r(x, y)$ extending $p(x) \cup q(y)$.
2. T is NIP iff for any definable pairwise commuting measures $\mu(x), \nu(y)$, $\varphi(x, y)$ and $\varepsilon > 0$, $\mu \otimes \nu(\varphi(x, y) \Delta \psi(x, y)) < \varepsilon$ for some $\psi(x, y)$ a Boolean combination of $\psi_i(x), \psi'_i(y)$.
3. n-tame: any relation $\varphi(x_1, \ldots, x_{n+1})$ can be "approximated" by relations
4. n-ary implies n-tame for any tameness (1-ary should imply distal - but there are no truly unary theories because of "=").

N-dependence

We fix a complete theory T in a language $\mathcal{L}$. For $k \geq 1$ we define:

- A formula $\varphi(x; y_1, \ldots, y_k)$ is *k-dependent* if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of T such that $A = \prod_{i=1}^{n} A_i$ is shattered by φ, where "A shattered" means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t.
$\mathcal{M} \models \varphi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$.
- T is *k-dependent* if all formulas are k-dependent.
- T is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent.
- 1-dependent = NIP $\subsetneq$ 2-dependent $\subsetneq \ldots$, as witnessed e.g. by the theory of the random k-hypergraph.

Examples of n-dependent structures

Theorem.[C., Hempel] If the field K is NIP, then the theory T of alternating n-linear forms over K (generalizing Granger) is (strictly) n-dependent.

(And if $K \models ACF$, then T is NSOP_1, essentially by the same proof as for $n = 2$ in [C., Ramsey]).

Theorem [Composition Lemma] Let $\mathcal{M}$ be an $\mathcal{L}'$-structure such that its reduct to a language $\mathcal{L} \subseteq \mathcal{L}'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \ldots, x_d)$ be an $\mathcal{L}$-formula, and $(y_0, \ldots, y_k)$ be arbitrary $k + 1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \ldots, i_k^t \leq k$ be arbitrary, and let $f_t : M_{y_{i_1^t}} \times \ldots \times M_{y_{i_k^t}} \to M_{x_t}$ be an arbitrary $\mathcal{L}'$-definable k-ary function. Then the formula

$$\psi(y_0; y_1, \ldots, y_k) := \varphi\left(f_1(y_{i_1^1}, \ldots, y_{i_k^1}), \ldots, f_d(y_{i_1^d}, \ldots, y_{i_k^d})\right)$$

is k-dependent.

Our earlier proof for $k = 2$ used a type counting criterion for types over infinite indiscernible sequences, and set-theoretic absoluteness. We have an analogous result for OP_2. Also for FOP_2 by Abd Aldaim, Conant, Terry.

Proof of the Composition Lemma, 1

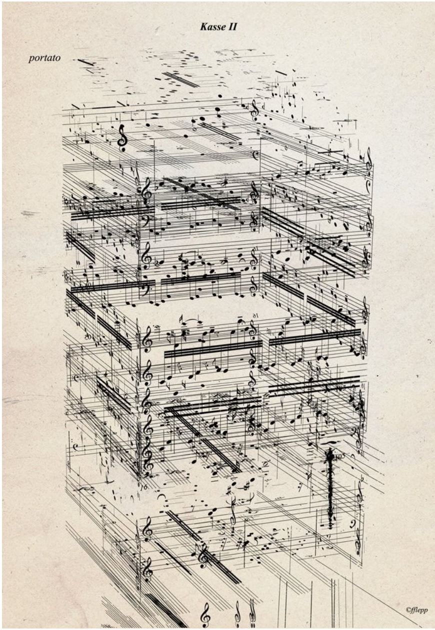
- Given a formula $\varphi(x; y_1, \ldots, y_k)$, $\varepsilon \in \mathbb{R}_{>0}$ and a function $f : \mathbb{N} \to \mathbb{N}$, we consider the following condition.

  $(\dagger)_{f,\varepsilon}$ There exists some $n^* \in \mathbb{N}$ such that the following holds for all $n^* \leq n \leq m \in \mathbb{N}$: For any mutually indiscernible sequences $I_1, \ldots, I_k$ of finite length, with $I_i \subseteq \mathbb{M}_{y_i}$, $n = |I_1| = \ldots = |I_{k-1}|$, $m = |I_k|$, and $b \in \mathbb{M}_x$ an arbitrary tuple there exists an interval $J \subseteq I_k$ with $|J| \geq \frac{m}{f(n)} - 1$ satisfying $|S_{\varphi,J}(b, I_1, \ldots, I_{k-1})| < 2^{n^{k-1-\varepsilon}}$.

- **Proposition.** The following are equivalent for a formula $\varphi(x; y_1, \ldots, y_k)$, with $k \geq 2$:
  1. $\varphi(x; y_1, \ldots, y_k)$ is k-dependent.
  2. There exist some $\varepsilon > 0$ and $d \in \mathbb{N}$ such that φ satisfies $(\dagger)_{f,\varepsilon}$ with respect to the function $f(n) = n^d$.
  3. There exist some $\varepsilon > 0$ and some function $f : \mathbb{N} \to \mathbb{N}$ such that φ satisfies $(\dagger)_{f,\varepsilon}$.

- This type-counting criterion can then be used to obtain some combinatorial stabilization of shattering on indiscernible arrays:

Proof of the Composition Lemma, 2



(“Kasse II, portato” by Frank Lepold)

Examples of n-dependent structures

In some sense all known "algebraic" examples are built from multilinear forms over NIP fields, is there some general theorem like this?

- [Cherlin-Hrushovski] smoothly approximable structures are 2-dependent: coordinatizable by bilinear forms / finite fields,
- infinite extra-special p-groups, and strictly n-dependent pure groups constructed using Mekler's construction [C., Hempel], using Baudisch's interpretation in alternating bilinear maps. Also generic n-nilpotent groups of odd prime exponent p, d'Elbée, Müller, Ramsey, Siniora.
- **Speculation.** If T is n-dependent, then it is "linear, or 1-based" relative to its NIP part.
- **Conjecture.** If K is an n-dependent field (pure, or with valuation, derivation, etc.), then K is NIP.
- Mounting evidence: n-dependent fields are Artin-Schreier closed (Hempel), valued char p are Henselian (C., Hempel), for valued fields reduces to pure fields (Boissonneau),...

Higher amalgamation, 1

Higher amalgamation was studied by a number of authors, starting with Shelah's work on stability in AEC's, Hrushovski in the study of the saturation spectrum and of generalized imaginaries , continued in a series of papers by Goodrick, Kim, Kolesnikov and others...

Definition

For $n \in \omega$, let $[n] = \{1, \ldots, n\} \in \omega$. For a set X, we let $\mathcal{P}(X)$ be the set of all subsets of X, $\mathcal{P}_{<n}(X)$ ($\mathcal{P}_{\leq n}(X)$) the set of all subsets of X of size less (respectively, less or equal) than n, and $\mathcal{P}^{-}(X) := \mathcal{P}(X) \setminus \{X\}$. For $s \subseteq X$, we let $(\downarrow s) := \{t \subseteq X : t \subseteq s\}$.

We let T be a complete *simple* first-order theory in a language $\mathcal{L}$, and we work in $\mathbb{M}^{\mathrm{heq}}$, the expansion of $\mathbb{M}$ by the hyper-imaginaries. As usual, $\perp$ denotes forking independence, $\perp^{u}$ denotes finite satisfiability, and $bdd(A)$ is the bounded closure of the set A in $\mathbb{M}^{\mathrm{heq}}$.

Higher amalgamation, 2

Definition

Let X be an arbitrary small set, and $S \subseteq \mathcal{P}(X)$ be non-empty and closed under subsets (so in particular $\emptyset \in S$). Let $\{r_s(x_s) : s \in S\}$ be a family of complete types over $\emptyset$ (where each x_s is a possibly infinite tuple of variables). We say that such a family of types is *independent* if:

1. if $a_\emptyset \models r_\emptyset$, then the set of elements of the tuple $a_\emptyset$ is boundedly closed;
2. if $s, t \in S$ and $s \subsetneq t$, then $x_s \subsetneq x_t$ and $r_s \subsetneq r_t$;
3. for all $s, t \in S$ we have $x_s \cap x_t = x_{s \cap t}$;
4. if $s \in S$ and $a_s \models r_s$, then:
   4.1 the set $\left\{a_{\{t\}} : t \in S\right\}$ is independent over $a_\emptyset$, where $a_{\{t\}}$ is a subtuple of a_s corresponding to the subtuple of the variables $x_{\{t\}} \subseteq x_s$;
   4.2 the set of elements of the tuple a_s is equal to $bdd\left(\bigcup_{t \in S} a_{\{t\}}\right)$, and the map $a_s \to x_s$ between the realizations and the variables is a bijection.

Higher amalgamation, 3

Definition

1. For $n \geq 1$, T satisfies (*independent*) *n-amalgamation* if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists a complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.
2. T satisfies (*independent*) *n-uniqueness* if for every independent system of types $\{r_s(x_s) : s \in \mathcal{P}^-([n])\}$ there exists at most one complete type $r_n(x_n)$ such that $\{r_s(x_s) : s \in \mathcal{P}([n])\}$ is an independent system of types.
3. T satisfies n-amalgamation (n-uniqueness) *over a set* $A \subseteq \mathbb{M}$ if (1) (respectively, (2)) holds for every independent system of types with $r_\emptyset = \text{tp}(bdd(A))$.
4. T satisfies *complete n-amalgamation* (or $\leq$ *n-amalgamation*) if T satisfies m-amalgamation for all $1 \leq m \leq n$.

Higher amalgamation, 4

Lemma

Assume $n \geq 1$ and T has $(\leq n)$-amalgamation. Assume that X is a set, $s^* \in \mathcal{P}(X)$, $S \subseteq \mathcal{P}_{<n}(X)$ is non-empty and closed under subsets (and if $n = 1$, also that $X = \bigcup \{s : s \in (\downarrow s^*) \cup S\}$), so that $\{r_s(x_s) : s \in (\downarrow s^*) \cup S\}$ is an independent system of types. Then $\{r_s(x_s) : s \in (\downarrow s^*) \cup S\}$ can be extended to an independent system of types $\{r_s(x_s) : s \in \mathcal{P}(X)\}$.

Problem

Is analogous statement true in $NSOP_1$ theories, with forking independence replaced by Kim-independence? Note that we have used base monotonicity of forking in the proof.

Higher stationarity and n-dependence

Theorem

Given $n \geq 1$, let T be a simple theory with $\leq (n+2)$-amalgamation (over models). Then T is n-dependent if and only if T has $(n+1)$-uniqueness (over models).

For $n = 1$ this corresponds to the well-known fact that if T is simple (hence satisfies ≤ 3-amalgamation over models) and there exists a non-stationary type (i.e. 2-stationarity fails), then T is not NIP.

Definition (Takeuchi)

A partitioned formula $\varphi(x; y_1, y_2)$ has OP_2 (probably not the final name) if there exist sequences $(a_i)_{i \in \omega}, (b_j)_{j \in \omega}$ with $a_i \in \mathbb{M}^{y_1}, b_j \in \mathbb{M}^{y_2}$ so that for every strictly increasing $f : \omega \to \omega$ there exists $c_f \in \mathbb{M}^x$ satisfying $\models \varphi(c_f, a_i, b_j) \iff i \leq f(j)$ for all $(i, j) \in \omega^2$.

A related property FOP_2 with increasing functions replaced by arbitrary functions $f : \omega \to \omega$ was also considered by Takeuchi, and it was studied more recently by Terry and Wolf.

Further notions of binarity

We let $\mathcal{C} := (\mathbb{L}, C)$ be the *generic countable binary branching C-relation*, i.e. the Fraïssé limit of all finite binary branching C-relations. We also let $\mathcal{C}_{\prec} := (\mathbb{L}, C, \prec)$ be the *generic countable convexly ordered binary branching C-relation*, i.e. the Fraïssé limit of all finite convexly ordered binary branching C-relations.

Definition

A theory T is *$\mathcal{C}$-less* if there is no formula $\varphi(x, y, z)$ and $(a_g : g \in \mathbb{L})$ such that $\models \varphi(a_f, a_g, a_h) \iff \mathcal{C} \models C(f, g, h)$. Equivalently, if every $\mathcal{C}_{\prec}$-indiscernible is already $(\mathbb{L}, \prec)$-indiscernible.

Related to treeless theories considered by Kaplan, Ramsey, Simon (probably the same).

Theorem

$\mathcal{C}$-less theories form a proper subclass of NOP_2 theories (and more precisely, every $\mathcal{C}$-less formula is NOP_2).

Collapse of various binarities

Theorem

If T is simple with ≤ 4-amalgamation, then the following are equivalent:

1. *T satisfies 3-uniqueness;*
2. *T is 2-dependent;*
3. *T has no OP_2;*
4. *T has no FOP_2;*
5. *T is $\mathcal{C}$-less.*

- E.g., as bilinear forms over finite fields have a simple theory and satisfy n-amalgamation for all n, it follows that they are $\mathcal{C}$-less.