Model theory of multilinear forms

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Non-degeneracy of bilinear forms

- Let V be a vector space over a field K.
- A bilinear form $\langle -, - \rangle : V^2 \to K$ is *degenerate* if there exists a vector $v \in V, v \neq 0$ such that $\langle v, w \rangle = 0$ for all $w \in V$.
- If V has finite dimension, a bilinear form $\langle -, - \rangle$ is non-degenerate if and only if it is a *perfect pairing*, i.e. the maps $V \to V^*,\ v \mapsto \langle v, - \rangle$ and $V \to V^*,\ v \mapsto \langle -, v \rangle$ are isomorphisms.
- In other words, for any basis $v_1, \ldots, v_n$ of V and any $k_1, \ldots, k_n \in K$ there is $w \in V$ such that $\langle v_i, w \rangle = k_i$ for all $i = 1, \ldots, n$.
- A "local" version holds in infinite dimensional spaces: the bilinear form $\langle -, - \rangle$ is non-degenerate if and only if for any $m \in \mathbb{N}$, any linearly independent vectors $v_1, \ldots, v_m$ in V and any $k_1, \ldots, k_m \in K$ there is $w \in V$ such that $\langle v_i, w \rangle = k_i$ for all $i = 1, \ldots, m$.

Towards non-degeneracy of n-linear forms, 1

- A naive attempt to generalize non-degeneracy to n-linear forms $\langle -, \dots, - \rangle_n : V^n \to K$ would be: for any non-zero $v_1, \dots, v_{n-1} \in V$ there is $w \in V$ such that $\langle v_1, \dots, v_n, w \rangle \neq 0$.
- However, this condition typically cannot be satisfied under additional requirements, like alternation: we have for example that $\langle v, v, v_3, \dots, v_{n-1}, w \rangle_n = 0$ regardless of the choice of $v, v_3, \dots, v_{n-1}, w \in V$.
- To circumvent this issue, we work in the tensor product space $\bigotimes^{n-1} V$ modulo the subspace N of $\bigotimes^{n-1} V$ generated by the elements $v_1 \otimes \dots \otimes v_{n-1}$ for which the map $V \to K,\ w \mapsto \langle v_1, \dots, v_{n-1}, w \rangle$ should be the zero map.

Towards non-degeneracy of n-linear forms, 2

- For example, for alternating n-linear forms, we take the subspace N to be

$$\mathrm{Alt} := \mathrm{Span}\left(\{v_1 \otimes \ldots \otimes v_{n-1} \mid v_1, \ldots, v_{n-1} \text{ are lin. dep.}\}\right).$$

- For symmetric n-linear forms we let N be

$$\mathrm{Sym} := \mathrm{Span}\Big(\{v_1 \otimes \ldots \otimes v_{n-1} - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n-1)} \mid \sigma \in \mathrm{Sym}\left(\{1, \ldots, n-1\}\right)\}\Big).$$

- Then $\left(\bigotimes^{n-1} V\right) / \mathrm{Alt} \; = \bigwedge^{n-1} V$,i.e. the $(n-1)$th exterior power of V, and
- $\left(\bigotimes^{n-1} V\right) / \mathrm{Sym} \; = \bigvee^{n-1} V$, i.e. the $(n-1)$th symmetric power of V.

Towards non-degeneracy of n-linear forms, 3

- Any n-linear form $\langle -, \ldots, - \rangle_n$ on V gives rise to a bilinear form $\langle -, - \rangle_2$ on $\left(\bigotimes^{n-1} V\right) \times V$ defined by

$$\langle v_1 \otimes \ldots \otimes v_{n-1}, v \rangle_2 := \langle v_1, \ldots, v_{n-1}, v \rangle_n.$$

- We say that an n-linear form $\langle -, \ldots, - \rangle_n$ on V is *of type N* if $t/N = s/N$ in $\left(\bigotimes^{n-1} V\right)/N$ implies that $\langle t, v \rangle_2 = \langle s, v \rangle_2$ for all $v \in V$.
- In this case we refer to the pair $(V, \langle -, \ldots, - \rangle_n)$ as an *n-linear space of type N*. For such a space the associated bilinear form $\langle -, - \rangle_2$ is well-defined on $\left(\left(\bigotimes^{n-1} V\right)/N\right) \times V$.

Non-degeneracy of n-linear forms

An n-linear space $(V, \langle -, \dots, - \rangle_n)$ of type N is:

- *non-degenerate* if for any non-zero $t \in \left(\bigotimes^{n-1} V\right)/N$ there is $w \in V$ such that $\langle t, w \rangle_2 \neq 0$;
- a *perfect pairing* if the maps
  $V \to \left(\left(\bigotimes^{n-1} V\right)/N\right)^*, \; v \mapsto \langle -, v \rangle_2$ and
  $\left(\bigotimes^{n-1} V\right)/N \to V^*, \; t \mapsto \langle t, - \rangle_2$
  are vector space isomorphisms;
- *generic* if for any $m \in \mathbb{N}$ and any linearly independent elements $t_1, \dots, t_m \in \left(\bigotimes^{n-1} V\right)/N$ and $k_1, \dots, k_m \in K$ there is $w \in V$ such that $\langle t_i, w \rangle_2 = k_i$ for all $i = 1, \dots, m$.
- Note: any perfect pairing is generic.

Non-degeneracy of n-linear forms, 2

- **Lemma.** Let $(V, \langle -, \ldots, - \rangle_n)$ be an n-linear space with V of infinite dimension. Then $\langle -, \ldots, - \rangle_n$ is non-degenerate if and only if $\langle -, \ldots, - \rangle_n$ is generic.
- For an infinite dimensional vector space V, if the dimension of $\left(\bigotimes^{n-1} V\right)/N$ is at least as big as the dimension of V, which is the case for Alt and Sym, then an n-linear form on V can never be a perfect pairing.
- Let V be of dimension $d \in \mathbb{N} < \infty$, then all three notions coincide. If $n > 2$ and $d \neq n$ (respectively, $d \neq 1$), an n-linear form of type Alt (respectively, Sym) cannot be non-degenerate (for dimensional reasons). Thus, in contrast to the bilinear case $n = 2$, for $n > 2$ there are no non-degenerate n-linear forms of type Alt or Sym on vector spaces of dimension greater than n.

Non-degenerate n-linear forms exist

- **Lemma.** For any n-linear space $(U, \langle -, \ldots, - \rangle_n)$ of type N there is a vector space V of dimension at most $\aleph_0 + \dim(U)$ containing U and an n-linear form $[-, \ldots, -]_n$ on V of type N extending $\langle -, \ldots, - \rangle_n$ and such that $(V, [-, \ldots, -]_n)$ is non-degenerate.

N-linear forms as first-order structures

- We consider n-linear spaces as structures in the language $\mathcal{L}$ consisting of two sorts V and K, the ring language on K, the vector space language on V, scalar multiplication function $K \times V \to V$ and a function symbol $\langle -, \ldots, - \rangle_n$ for an n-linear form $V^n \to K$.
- The language $\mathcal{L}_{\theta,f}$ is obtained from $\mathcal{L}$ by adding:
  - for each $p \in \omega$ a p-ary predicate $\theta_p(v_1, \ldots, v_p)$ which holds if and only if $v_1, \ldots, v_p \in V$ are linearly independent over K;
  - for each $p \in \omega$ and $i \leq p$, a $(p+1)$-ary function symbol $f_i^p : V^{p+1} \to K$ interpreted as: $f_i^p(v; v_1, \ldots, v_p) = \lambda_i$ if $\models \theta_p(v_1, \ldots, v_p)$ and $v = \sum_{i=1}^{p} \lambda_i v_i$ for some $\lambda_1, \ldots, \lambda_p \in K$; and 0 otherwise.
- Let $\mathcal{L}^K$ be an expansion of the language of rings by relations on $K^p, p \in \omega$ definable in the language of rings such that K eliminates quantifiers in $\mathcal{L}^K$ (can always take Morleyzation of K).
- Let $\mathcal{L}_{\theta,f}^K := \mathcal{L}_{\theta,f} \cup \mathcal{L}^K$.

Quantifier elimination for non-degenerate n-linear forms

- Let $T := T^K_{n,N}$ be the theory of infinite dimensional non-degenerate n-linear spaces of type N, with the field sort a model of $\mathrm{Th}(K)$, in the language $\mathcal{L}^K_{\theta,f}$ (it is consistent — as every n-linear form extends to a non-degenerate one).
- **Proposition.** The set of partial $\mathcal{L}^K_{\theta,f}$-isomorphisms between two ω-saturated non-degenerate n-linear spaces of type Alt (over elementarily equivalent fields) has the back-and-forth property (and is non-empty).
- **Theorem.** The theory $T^K_{n,\mathrm{Alt}}$ of infinite dimensional non-degenerate n-linear spaces of type Alt over K has quantifier elimination (in the language $\mathcal{L}^K_{\theta,f}$) and is complete.
- For $n = 2$ is essentially due to Granger. The necessity of adding the functions f^p_i for QE was missed in Granger's work, and pointed out by D. MacPherson.
- In the symmetric case, some assumptions on the field K are needed (e.g. closure under square roots, in the case $n = 2$).

N-dependence

We fix a complete theory T in a language $\mathcal{L}$. For $k \geq 1$ we define:

- A formula $\varphi(x; y_1, \ldots, y_k)$ is *k-dependent* if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \ldots, k\}$ in a model $\mathcal{M}$ of T such that $A = \prod_{i=1}^{n} A_i$ is shattered by φ, where "A shattered" means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t.
$\mathcal{M} \models \varphi(b_s; a_{1,j_1}, \ldots, a_{k,j_k}) \iff (j_1, \ldots, j_k) \in s$.
- T is *k-dependent* if all formulas are k-dependent.
- T is *strictly k-dependent* if it is k-dependent, but not $(k-1)$-dependent.
- 1-dependent = NIP $\subsetneq$ 2-dependent $\subsetneq$ ..., as witnessed e.g. by the theory of the random k-hypergraph.

N-dependent theories

All known "algebraic" n-dependent examples come from bilinear forms over NIP fields:

- [Cherlin-Hrushovski] smoothly approximable structures are 2-dependent, and coordinatizable via bilinear forms over finite fields,
- infinite extra-special p-groups, and strictly n-dependent pure groups constructed using Mekler's construction [C., Hempel] are essentially of this form as well, using Baudisch's interpretation in alternating bilinear maps.
- **Speculation.** If T is n-dependent, then it is "linear, or 1-based" relative to its NIP part.
- **Conjecture.** If K is an n-dependent field (pure, or with valuation, derivation, etc.), then K is NIP.
- Mounting evidence: n-dependent fields are Artin-Schreier closed (Hempel), valued char p are Henselian (C., Hempel), for valued fields reduces to pure fields (Boissonneau),...

N-dependence of n-linear forms

- **Theorem.** If the field K is NIP, then $T^K_{n,\mathrm{Alt}}$ is (strictly) n-dependent.
- (And if $K \models ACF$, then $T^K_{n,\mathrm{Alt}}$ is NSOP_1, essentially by the same proof as for $n = 2$ in [C., Ramsey].)
- By QE and analysis of generalized indiscernibles, the proof that $T^K_{n,\mathrm{Alt}}$ is n-dependent reduces to showing that the composition of a relation definable in an NIP structure with *arbitrary* k-ary functions is k-dependent:

Composition Lemma

- **Theorem [Composition Lemma]** Let $\mathcal{M}$ be an $\mathcal{L}'$-structure such that its reduct to a language $\mathcal{L} \subseteq \mathcal{L}'$ is NIP. Let $d, k \in \mathbb{N}$, $\varphi(x_1, \ldots, x_d)$ be an $\mathcal{L}$-formula, and $(y_0, \ldots, y_k)$ be arbitrary $k+1$ tuples of variables. For each $1 \leq t \leq d$, let $0 \leq i_1^t, \ldots, i_k^t \leq k$ be arbitrary, and let $f_t : M_{y_{i_1^t}} \times \ldots \times M_{y_{i_k^t}} \to M_{x_t}$ be an arbitrary $\mathcal{L}'$-definable k-ary function. Then the formula

$$\psi\left(y_0; y_1, \ldots, y_k\right) := \varphi\left(f_1(y_{i_1^1}, \ldots, y_{i_k^1}), \ldots, f_d(y_{i_1^d}, \ldots, y_{i_k^d})\right)$$

  is k-dependent.
- Our earlier proof for $k = 2$ used a certain type counting criterion for types over infinite indiscernible sequences, and set-theoretic absoluteness.

Proof of the Composition Lemma, 1

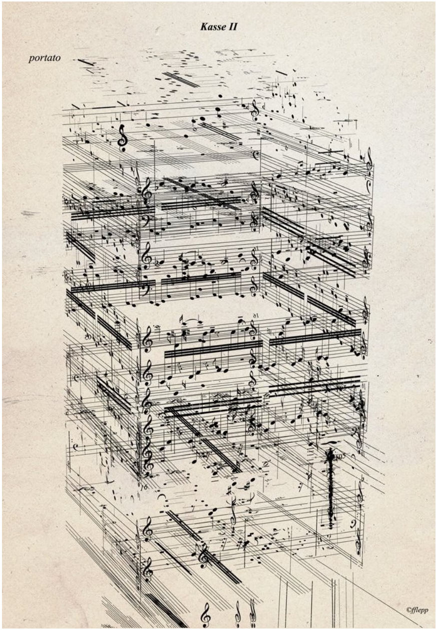
- Given a formula $\varphi(x; y_1, \ldots, y_k)$, $\varepsilon \in \mathbb{R}_{>0}$ and a function $f : \mathbb{N} \to \mathbb{N}$, we consider the following condition.

  $(\dagger)_{f,\varepsilon}$ There exists some $n^* \in \mathbb{N}$ such that the following holds for all $n^* \leq n \leq m \in \mathbb{N}$: For any mutually indiscernible sequences $I_1, \ldots, I_k$ of finite length, with $I_i \subseteq \mathbb{M}_{y_i}$, $n = |I_1| = \ldots = |I_{k-1}|$, $m = |I_k|$, and $b \in \mathbb{M}_x$ an arbitrary tuple there exists an interval $J \subseteq I_k$ with $|J| \geq \frac{m}{f(n)} - 1$ satisfying $|S_{\varphi,J}(b, I_1, \ldots, I_{k-1})| < 2^{n^{k-1-\varepsilon}}$.

- **Proposition.** The following are equivalent for a formula $\varphi(x; y_1, \ldots, y_k)$, with $k \geq 2$:
  1. $\varphi(x; y_1, \ldots, y_k)$ is k-dependent.
  2. There exist some $\varepsilon > 0$ and $d \in \mathbb{N}$ such that φ satisfies $(\dagger)_{f,\varepsilon}$ with respect to the function $f(n) = n^d$.
  3. There exist some $\varepsilon > 0$ and some function $f : \mathbb{N} \to \mathbb{N}$ such that φ satisfies $(\dagger)_{f,\varepsilon}$.

- This type-counting criterion can then be used to obtain some combinatorial stabilization of shattering on indiscernible arrays:

Proof of the Composition Lemma, 2



(“Kasse II, portato” by Frank Lepold)

Connected components G^{00} and G^{∞}

- Let T be a theory and G a type-definable group (over $\emptyset$), and $A \subseteq \mathbb{M}$ a small subset.
- Let G^{00}_A (resp., G^{∞}_A) be the smallest type-definable (resp., invariant) over A subgroup of G of bounded index.
- [Shelah, Gismatullin] If T is NIP, then $G^{00}_A = G^{00}_{\emptyset}$ and $G^{\infty}_A = G^{\infty}_{\emptyset}$ for all small A.
- **Example.** Let $G := \bigoplus_{\omega} \mathbb{F}_p$. Let $\mathcal{M} := (G, \mathbb{F}_p, 0, +, \cdot)$ with $\cdot$ the bilinear form $(a_i) \cdot (b_i) = \sum_i a_i b_i$ from G to $\mathbb{F}_p$.
- Then G is 2-dependent and $G^{00}_A = \left\{ g \in G : \bigcap_{a \in A} g \cdot a = 0 \right\}$ — gets smaller when enlarging A.
- However, for any A, B we have $G^{00}_{A \cup B} = G^{00}_A \cap G^{00}_B$.
- And for a non-degenerate n-linear form over $\mathbb{F}_p$ and any $A_1, \ldots, A_n$, $G^{00}_{A_1 \cup \ldots \cup A_n} = \bigcap_{i=1}^{n} G^{00}_{\bigcup_{j \neq i} A_j}$.

Connected components G^{00} and G^{∞} for n-dependent G

- **Theorem.** If T is n-dependent and $G = G(\mathbb{M})$ is a type-definable group (over $\emptyset$), then for any small model $\mathcal{M}$ and finite tuples $b_1, \ldots, b_{n-1}$ in $\mathbb{M}$ *sufficiently independent* over $\mathcal{M}$, we have

$$G^{00}_{\mathcal{M}\cup b_1 \cup \cdots \cup b_{n-1}} = \bigcap_{i=1,\ldots,n-1} G^{00}_{\mathcal{M}\cup b_1 \cup \ldots \cup b_{i-1} \cup b_{i+1} \cup \ldots \cup b_{n-1}} \cap G^{00}_{C \cup b_1 \cup \cdots \cup b_{n-1}}$$

  for some $C \subseteq \mathcal{M}$ of absolutely bounded size.
- This generalizes [Shelah] for $n = 1, 2$, where general position is not needed.
- So far, we can prove an analogous statement for G^{∞} when G is abelian.

"Sufficiently independent"

- (κ-*coheirs*) For a cardinal κ, any model $\mathcal{M}$, and any tuple a we write $a \mathop{\smile\!\!\!\!\!\!\mid}^{u,\kappa}_{\mathcal{M}} B$ if for any set $C \subset B \cup \mathcal{M}$ of size $< \kappa$, $\mathrm{tp}(a/C)$ is realized in $\mathcal{M}$.
- Let $\mathcal{M}$ be a small model, and $\bar{b}_1, \ldots, \bar{b}_{n-1}$ finite tuples in $\mathbb{M}$. We say that $(\mathcal{M}, \bar{b}_1, \ldots, \bar{b}_{n-1})$ are in a *generic position* if there exist regular cardinals $\kappa_1 < \kappa_2 < \ldots < \kappa_{n-1}$ and models $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \ldots \preceq \mathcal{M}_{n-1} = \mathcal{M}$ such that $\beth_2(|\mathcal{M}_i|)^+ \leq \kappa_{i+1}$ for $i = 0, \ldots, n-2$ and

$$\bar{b}_i \mathop{\smile\!\!\!\!\!\!\mid}^{u,\kappa_i}_{\mathcal{M}_i} \bar{b}_{<i}\mathcal{M}_{n-1}$$

  for all $1 \leq i \leq n-1$.
- Generic position can always be arranged using mutually indiscernible sequences / commuting global invariant types.
- We don't know if any assumption on the b_i at all is needed.

Thank you!

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