Recognizing groups and fields in Erdős geometry and model theory

Artem Chernikov

UCLA

Wilhelm Killing Kolloquium, Münster (via Zoom)

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Point-line incidences

1. Given n points and n lines in $\mathbb{R}^2$, how many incidences can there be?
2. Obvious upper bound n^2.
3. [Elekes'02] construction for a lower bound $\Omega\left(n^{\frac{4}{3}}\right)$:

Hypergraphs and Zarankiewicz's problem

- We fix $r \in \mathbb{N}_{\geq 2}$ and let $H = (V_1, \ldots, V_r; E)$ be an r-partite hypergraph of size n (or just *r-hypergraph*) with vertex sets $V_1, \ldots, V_r$ with $|V_i| = n$ and (hyper-)edge set $E \subseteq \prod_{i \in [r]} V_i$.
- When $r = 2$, we say "bipartite graph" instead of "2-hypergraph".
- For $k \in \mathbb{N}$, let $K_{k,\ldots,k}$ denote the complete r-hypergraph with each part of size k (i.e. $V_i = [k]$ and $E = \prod_{i \in [k]} V_i$).
- H is *$K_{k,\ldots,k}$-free* if it does note contain an isomorphic copy of $K_{k,\ldots,k}$.
- Zarankiewicz's problem: for fixed r, k, what is the maximal number of edges $|E|$ in a $K_{k,\ldots,k}$-free r-hypergraph H? (As a functions of n.)

Number of edges in a $K_{k,\ldots,k}$-free hypergraph

- The following fact is due to [Kővári, Sós, Turán'54] for $r = 2$ and [Erdős'64] for general r.

Fact (The Basic Bound)

If H is a $K_{k,\ldots,k}$-free r-hypergraph then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$.

- So the exponent is slightly better than the maximal possible r (we have n^r edges in $K_{n,\ldots,n}$). A probabilistic construction in [Erdős'64] shows that this bound cannot be substantially improved (but whether it is sharp up to a constant is widely open).
- Restricting to hypergraphs that are defined "geometrically", one might expect stronger bounds on the exponent.

Semialgebraic hypergraphs

- A set $X \subseteq \mathbb{R}^d$ is *semialgebraic* if X is a finite union of sets of the form

$$\left\{\bar{x} \in \mathbb{R}^d : f_1(\bar{x}) \geq 0, \ldots, f_p(\bar{x}) \geq 0, f_{p+1}(\bar{x}) > 0, \ldots, f_q(\bar{x}) > 0\right\},$$

  where $p \leq q \in \mathbb{N}$ and each $f_i \in \mathbb{R}[\bar{x}]$ is a polynomial in d variables.
- X has *(description) complexity* t if $d \leq t$, it is a union of at most t such sets, $q \leq t$ and $\deg(f_i) \leq t$ for all i.
- A finite r-hypergraph $H = (V_1, \ldots, V_r; E)$ is *semialgebraic, of complexity* t if $V_i \subseteq \mathbb{R}^{d_i}$ for some d_i and $E = \left(\prod_{i \in [r]} V_i\right) \cap X$ for some semialgebraic set $X \subseteq \mathbb{R}^{d_1 + \ldots + d_r}$ of complexity t (up to isomorphism).
- A lot of (hyper-)graphs arising in incidence combinatorics of elementary geometric shapes are semialgebraic, of small complexity.

Example: point-line incidences on the plane

- Let $I \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the incidence relation between points and lines on the plane, i.e.

$$I(x_1, x_2; y_1, y_2) \iff x_2 = y_1 x_1 + y_2.$$

- Then I is semialgebraic (of complexity 2) and $K_{2,2}$-free (for any two points belong to at most one line).
- Let V_1 be a set of n points and V_2 a set of n lines on the plane $\mathbb{R}^2$, and $E := I \restriction_{V_1 \times V_2}$. Then the bipartite graph $(V_1, V_2; E)$ satisfies the basic bound of Kővári, Sós, Turán:

$$|E| = O\left(n^{\frac{3}{2}}\right).$$

- While this is optimal for general graphs, utilizing the geometry of the reals:

Fact (Szémeredi-Trotter '83)

In fact, $|E| = O\left(n^{\frac{4}{3}}\right)$ *— matching the lower bound up to a constant.*

- Note that $\frac{4}{3} < \frac{3}{2}$.

Zarankiewicz for semialgebraic (hyper-)graphs

- Szémeredi-Trotter theorem has numerous generalizations for semialgebraic graphs, e.g. [Pach, Sharir'98], [Elekes, Szabó'12], and more generally

Fact (Fox, Pach, Sheffer, Suk, Zahl'17)

If $(V_1, V_2; E)$, with $V_i \subseteq \mathbb{R}^{d_i}$, is a semialgebraic bipartite graph of complexity t and $K_{k,k}$-free, then for any $\varepsilon > 0$,

$$|E| = O_{t,d_1,d_2,k,\varepsilon}\left(n^{\frac{2d_1d_2-d_1-d_2}{d_1d_2-1}+\varepsilon}\right).$$

- **Moral:** for semialgebraic graphs, the bound is of the form $O(n^{e-\varepsilon})$ for some $\varepsilon > 0$, where e is given by the basic bound for arbitrary graphs.
- Generalizations to semialgebraic hypergraphs [Do'18].

Connections to the "trichotomy principle" in model theory

- The *trichotomy principle* in model theory: in a sufficiently tame context (including semialgebraic), every structure is either "trivial", or essentially a vector space, or interprets a field (see below).
- In this talk: the exponents in Zarankiewicz bounds for semialgebraic (hyper-)graphs reflect the trichotomy principle, and detect presence of algebraic structures (groups, fields).
- Instances of this principle are also known in combinatorics — extremal configuration for various counting problems tend to come from algebraic structures. So here we discuss two "inverse" theorems which show this is the only way!

Elekes-Szabó theorem, 1

- [Erdős, Szemerédi'83] There exists some $c \in \mathbb{R}_{>0}$ such that: for every finite $A \subseteq \mathbb{R}$,

$$\max \{|A + A|, |A \cdot A|\} = \Omega \left(|A|^{1+c}\right).$$

- [Solymosi], [Konyagin, Shkredov] Holds with $\frac{4}{3} + \varepsilon$ for some sufficiently small $\varepsilon > 0$. (Conjecturally: with $2 - \varepsilon$ for any ε).
- [Elekes, Rónyai'00] Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree d, then for all $A, B \subseteq_n \mathbb{R}$,

$$|f(A \times B)| = \Omega_d \left(n^{\frac{4}{3}}\right),$$

unless f is either of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials g, h, i.

Elekes-Szabó theorem, 2

- [Elekes-Szabó'12] provide a conceptual generalization: for any algebraic surface $Q(x_1, x_2, x_3) \subseteq \mathbb{R}^3$ so that the projection onto any two coordinates is finite-to-one, exactly one of the following holds:
  1. there exists $\gamma > 0$ s.t. for any finite $A_i \subseteq_n \mathbb{R}$ we have
  $$|Q \cap (A_1 \times A_2 \times A_3)| = O(n^{2-\gamma}).$$
  2. There exist open sets $U_i \subseteq \mathbb{R}$ and $V \subseteq \mathbb{R}$ containing 0, and analytic bijections with analytic inverses $\pi_i : U_i \to V$ such that
  $$\pi_1(x_1) + \pi_2(x_2) + \pi_3(x_3) = 0 \Leftrightarrow Q(x_1, x_2, x_3)$$
  for all $x_i \in U_i$.

Generalizations of the Elekes-Szabó theorem

Let $Q \subseteq X_1 \times \ldots \times X_r$ be an algebraic surface with finite-to-one projection onto any $r-1$ coordinates and $\dim(X_i) = m$.

1. [Elekes, Szabó'12] $r = 3$, m arbitrary over $\mathbb{C}$ (only count on grids in *general position*, correspondence with a complex algebraic group of dimension m);
2. [Raz, Sharir, de Zeeuw'18] $r = 4$, $m = 1$ over $\mathbb{C}$;
3. [Raz, Shem-Tov'18] $m = 1$, Q of the form $f(x_1, ..., x_{r-1}) = x_r$ for f a polynomial and any r over $\mathbb{C}$.
4. [Bays, Breuillard'18] r and m arbitrary over $\mathbb{C}$, recognized that the arising groups are abelian (however no bounds on γ);
5. Related work: [Raz, Sharir, de Zeeuw'15], [Wang'15]; [Bukh, Tsimmerman' 12], [Tao'12]; [Hrushovski'13]; [Jing, Roy, Tran'19].
6. [C., Peterzil, Starchenko' 21] Any r and m, Q semialgebraic, explicit bounds on γ. A special case:

Theorem (C., Peterzil, Starchenko)

Assume $r \geq 3$ *and* $Q \subseteq \mathbb{R}^r$ *is semi-algebraic, of description complexity* d*, such that the projection of* Q *to any* $r-1$ *coordinates is finite-to-one. Then exactly one of the following holds.*

1. *For any finite* $A_i \subseteq_n \mathbb{R}$, $i \in [r]$, *we have*

$$|Q \cap (A_1 \times \ldots \times A_r)| = O_{r,d}\left(n^{r-1-\gamma}\right),$$

   where $\gamma = \frac{1}{3}$ *if* $r \geq 4$, *and* $\gamma = \frac{1}{6}$ *if* $r = 3$.

2. *There exist open sets* $U_i \subseteq \mathbb{R}, i \in [r]$, *an open set* $V \subseteq \mathbb{R}$ *containing* 0, *and analytic bijections with analytic inverses* $\pi_i : U_i \to V$ *such that*

$$\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \Leftrightarrow Q(x_1, \ldots, x_r)$$

   for all $x_i \in U_i, i \in [r]$.

General *o*-minimal case

Theorem (C., Peterzil, Starchenko)

Assume $r \geq 3$, $Q \subseteq X_1 \times \cdots \times X_r$ are ***definable in an o-minimal expansion of*** $\mathbb{R}$ *with* $\mathbf{dim}\,(\boldsymbol{X_i}) = \boldsymbol{m}$*, and the projection of Q to any $r-1$ coordinates is finite-to-one. Then* ~~***exactly***~~ *one of the following holds.*

1. *For any finite $A_i \subseteq_n X_i$* ***in general position****, $i \in [r]$, we have*

$$|Q \cap (A_1 \times \ldots \times A_r)| = O_Q\left(n^{r-1-\gamma}\right),$$

   for $\gamma = \frac{1}{8m-5}$ if $r \geq 4$, and $\gamma = \frac{1}{16m-10}$ if $r = 3$.

2. *There exist definable relatively open sets $U_i \subseteq X_i$,* ***an abelian Lie group*** $(G, +)$ ***of dimension*** *m and an open neighborhood $V \subseteq G$ of 0, and* ***definable homeomorphisms*** *$\pi_i : U_i \to V$, such that for all $x_i \in U_i, i \in [r]$*

$$\pi_1(x_1) + \cdots + \pi_r(x_r) = 0 \Leftrightarrow Q(x_1, \ldots, x_r).$$

Remarks

1. So Q can be defined not only using polynomial (in-)equalities, but also e.g. using e^x and restricted analytic functions.
2. One ingredient — improved Zarankiewicz bounds also hold in o-minimal structures ([Basu, Raz], [C., Galvin, Starchenko]). The power saving γ in the non-group case corresponds to the non-trivial improvement on the basic bound.
3. Another – a higher arity generalization of the Abelian Group Configuration theorem of Zilber and Hrushovski on recognizing groups from a "generic chunk". We discuss a simple purely combinatorial case:

Recognizing groups, 1

1. Assume that $(G, +, 0)$ is an abelian group, and consider the r-ary relation $Q \subseteq \prod_{i \in [r]} G$ given by $x_1 + \ldots + x_r = 0$.
2. Then Q is easily seen to satisfy the following two properties, for any permutation of the variables of Q:

$$\forall x_1, \ldots, \forall x_{r-1} \exists ! x_r Q(x_1, \ldots, x_r), \tag{P1}$$

$$\forall x_1, x_2 \forall y_3, \ldots y_r \forall y_3', \ldots, y_r' \Big(Q(\bar{x}, \bar{y}) \wedge Q(\bar{x}, \bar{y}') \rightarrow \tag{P2}$$
$$\big(\forall x_1', x_2' Q(\bar{x}', \bar{y}) \leftrightarrow Q(\bar{x}', \bar{y}')\big)\Big).$$

We show a converse, assuming $r \geq 4$:

Recognizing groups, 2

Theorem (C., Peterzil, Starchenko)

Assume $r \in \mathbb{N}_{\geq 4}$, $X_1, \ldots, X_r$ and $Q \subseteq \prod_{i \in [r]} X_i$ are sets, so that Q satisfies (P1) and (P2) for any permutation of the variables. Then there exists an abelian group $(G, +, 0_G)$ and bijections $\pi_i : X_i \to G$ such that for every $(a_1, \ldots, a_r) \in \prod_{i \in [r]} X_i$ we have

$$Q(a_1, \ldots, a_r) \iff \pi_1(a_1) + \ldots + \pi_r(a_r) = 0_G.$$

- If $X_1 = \ldots = X_r$, property (P1) is equivalent to saying that the relation Q is an $(r-1)$-*dimensional permutation* on the set X_1, or a *Latin* $(r-1)$-*hypercube*, as studied by Linial and Luria. Thus the condition (P2) characterizes, for $r \geq 3$, those Latin r-hypercubes that are given by the relation "$x_1 + \ldots + x_{r-1} = x_r$" in an abelian group.

Recognizing fields

- For the semialgebraic $K_{2,2}$-free point-line incidence relation $Q = \{(x_1, x_2; y_1, y_2) \in \mathbb{R}^4 : x_2 = y_1 x_1 + y_2\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ we have the (optimal) lower bound $|Q \cap (V_1 \times V_2)| = \Omega(n^{\frac{4}{3}})$.
- To define it we use both addition and multiplication, i.e. the field structure.
- This is not a coincidence — any non-trivial lower bound on the Zarankiewicz's exponent of Q allows to recover a field from it:

Theorem (Basit, C., Starchenko, Tao, Tran)

Assume that $Q \subseteq \mathbb{R}^d = \prod_{i \in [r]} \mathbb{R}^{d_i}$ for some $r, d_i \in \mathbb{N}$ is definable in an o-minimal structure and $K_{k,\ldots,k}$-free, but $|Q \cap \prod_{i \in [r]} V_i| \neq O(n^{r-1})$. Then a real closed field is definable in the first-order structure $(\mathbb{R}, <, Q)$.

Ingredients

- An almost optimal Zarankiewicz bound for hypergraphs definable in *locally modular* *o*-minimal expansions of groups, so e.g. for *semilinear* (i.e. defined using *linear* (in-)equalities) hypergraphs.
- The trichotomy theorem for *o*-minimal structures from model theory [Peterzil, Starchenko].

A matroid associated to an o-minimal structure

- Given a structure M, $A \subseteq M$ and a finite tuple a in M, $a \in \mathrm{acl}(A)$ if it belongs to some finite A-definable subset of $M^{|a|}$ (this generalizes linear span in vector spaces and algebraic closure in fields).
- $\dim(a/A)$ is the minimal cardinality of a subtuple a' of a so that $\mathrm{acl}(a \cup A) = \mathrm{acl}(a' \cup A)$ (in an algebraically closed field, this is just the transcendence degree of a over the field generated by A).
- Given a finite tuple a and sets $C, B \subseteq M$, we write $a \downarrow_C B$ to denote that $\dim(a/BC) = \dim(a/C)$.
- In an o-minimal structure, $\downarrow$ is a well-behaved notion of independence defining a *matroid*.

Local modularity

- An o-minimal structure is *(weakly) locally modular* if for any small subsets $A, B \subseteq \mathbb{M} \models T$ there exists some small set $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(AC)\cap\mathrm{acl}(BC)} B$.
- Intuition: the algebraic closure operator behaves like the linear span in a vector space, as opposed to the algebraic closure in an algebraically closed field.
- In particular, an o-minimal structure is locally modular if and only if any normal interpretable family of plane curves in T has dimension ≤ 1.

Zarankiewicz bound for semilinear relations

Theorem (Basit, C., Starchenko, Tao, Tran)

Let $\mathcal{M}$ be an o-minimal locally modular expansion of a group and Q a definable relation of arity $r \geq 2$. Then for any $\varepsilon > 0$ and any V_i with $|V_i| = n$ such that $E := Q \cap V_1 \times \ldots \times V_r$ is $K_{k,\ldots,k}$-free, we have

$$|E| = O_{Q,k,\varepsilon}\left(n^{r-1+\varepsilon}\right).$$

Moreover, if Q itself is $K_{k,\ldots,k}$-free, then for any V_i with $|V_i| = n$ we have

$$|E| = O_Q(n^{r-1}).$$

Recovering a field in the *o*-minimal case

Fact (Peterzil, Starchenko'98)

Let $\mathcal{M}$ be an o-minimal (saturated) structure. TFAE:

- *$\mathcal{M}$ is not locally modular;*
- *there exists a real closed field definable in $\mathcal{M}$.*

- [Marker, Peterzil, Pillay'92] Let $X \subseteq \mathbb{R}^n$ be a semialgebraic but not semilinear set. Then $\cdot \restriction_{[0,1]^2}$ is definable in $(\mathbb{R}, <, +, X)$. In particular, it is not locally modular.
- Combining this with the optimal bound in the locally modular case, we get the result.

Extra: corollary for semilinear hypergraphs

Corollary

For every $r, s, k \in \mathbb{N}$ there exist some $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta(r, s) := s(2^{r-1} - 1)$ satisfying the following. Suppose $r \geq 2, d = d_1 + \ldots + d_r \in \mathbb{N}$ and $Q \subseteq \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_r}$ is semilinear, defined by $\leq s$ linear (in-)equalities. Then for any $V_i \subseteq_n \mathbb{R}^{d_i}$ so that $E := Q \cap \prod_{i \in [r]} V_i$ is $K_{k,\ldots,k}$-free we have

$$|E| \leq \alpha n^{r-1} (\log n)^{\beta} .$$

- **Example.** For any set V_1 of n points and any set V_2 of n (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k}\left(n(\log n)^{2d}\right)$ incidences.

Problem

We show that the logarithmic factor is unavoidable. But what is the optimal power of $\log n$? In particular, does it depend on d?

Thank you!

- *Model-theoretic Elekes-Szabó for stable and o-minimal hypergraphs*, Artem Chernikov, Ya'acov Peterzil, Sergei Starchenko (arXiv:2104.02235)
- *Zarankiewicz's problem for semilinear hypergraphs*, Artem Chernikov, Abdul Basit, Sergei Starchenko, Terence Tao and Chieu-Minh Tran (arXiv:2009.02922)