Idempotent Keisler measures

Artem Chernikov
(joint with Kyle Gannon)

UCLA

Logic Seminar at Imperial College, London, UK (via Zoom)

May 19, 2021

Spaces of types

- Let T be a complete first-order theory in a language $\mathcal{L}$, $\mathbb{M} \models T$ a monster model (i.e. κ-saturated and κ-homogeneous for a sufficiently large cardinal κ), $\mathcal{M} \preceq \mathbb{M}$ a small elementary submodel.
- For $A \subseteq \mathbb{M}$ and x an arbitrary tuple of variables, $S_x(A)$ denotes the set of complete types over A.
- Let $\mathcal{L}_x(A)$ denote the set of all formulas $\varphi(x)$ with parameters in A, up to logical equivalence — which we identify with the Boolean algebra of A-definable subsets of $\mathbb{M}_x$; $\mathcal{L}_x := \mathcal{L}_x(\emptyset)$.
- Then the types in $S_x(A)$ are the ultrafilter on $\mathcal{L}_x(A)$.
- By Stone duality, $S_x(A)$ is a totally disconnected compact Hausdorff topological space with a basis of clopen sets of the form
$$\langle\varphi\rangle := \{p \in S_x(A) : \varphi(x) \in p\}$$
for $\varphi(x) \in \mathcal{L}_x(A)$.
- We refer to types in $S_x(\mathbb{M})$ as *global types*.

Keisler measures

- A *Keisler measure* μ in variables x over $A \subseteq \mathbb{M}$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of A-definable subsets of $\mathbb{M}_x$.
- $\mathfrak{M}_x(A)$ denotes the set of all Keisler measures in x over A.
- Then $\mathfrak{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

  with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A), r_i, s_i \in [0,1]$ for $i < n$.
- Identifying p with the Dirac measure δ_p, $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathfrak{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$.

Some examples of Keisler measures

1. In arbitrary T, given $p_i \in S_x(A)$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ with $\sum_{i \in \mathbb{N}} r_i = 1$, $\mu := \sum_{i \in \mathbb{N}} r_i \delta_{p_i} \in \mathfrak{M}_x(A)$.
2. Let $T = \mathrm{Th}(\mathbb{N}, =)$, $|x| = 1$. Then

$$S_x(\mathbb{M}) = \{\mathrm{tp}(a/\mathbb{M}) : a \in \mathbb{M}\} \cup \{p_\infty\},$$

   where p_∞ is the unique non-realized type axiomatized by $\{x \neq a : a \in \mathbb{M}\}$. By QE, every formula is a Boolean combination of $\{x = a : a \in \mathbb{M}\}$, from which it follows that every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is as in (1).
3. More generally, if T is ω-stable (e.g. strongly minimal, say ACF_p for p prime or 0) and x is finite, then every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is a sum of types as in (1).
4. Let $T = \mathrm{Th}(\mathbb{R}, <)$, λ be the Lebesgue measure on $\mathbb{R}$ and $|x| = 1$. For $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, define $\mu(\varphi) := \lambda\left(\varphi(\mathbb{M}) \cap [0,1]_{\mathbb{R}}\right)$ (this set is Borel by QE). Then $\mu(X)$ is a Keisler measure, but not a sum of types as in (1).

Brief history of the theory of Keisler measures

- Measures and forking in stable/NIP theories [Keisler'87]
- Automorphism-invariant measures in ω-categorical structures [Albert'92, Ensley'96]
- Applications to neural networks [Karpinski, Macyntire'00]
- Pillay's conjecture and compact domination [Hrushovski, Peterzil, Pillay'08], [Hrushovski, Pillay'11], [Hrushovski, Pillay, Simon'13]
- Randomizations [Ben Yaacov, Keisler'09] (NIP and stability are preserved)
- Approximate Subgroups [Hrushovski'12]
- Definably amenable NIP groups [C., Simon'15] (in particular translation-invariant measures are classified)
- Tame (equivariant) regularity lemmas: subsets of [C., Conant, Malliaris, Pillay, Shelah, Starchenko, Terry, Tao, Towsner, ...'11– ...]
- See my review "Model theory, Keisler measures and groups", The Bulletin of Symbolic Logic, 24(3), 336-339 (2018)

Keisler measures outside of NIP?

- All of the above — mostly inside the context of NIP theories (thanks to the *VC-theory*, measures are strongly approximated by types).
- Pseudofinite fields — ultraproducts of finite counting measures are very well-behaved (more generally in *MS-measurable* structures).
- But very few general results outside of NIP so far. Some counterexamples:
  - Independent product $\otimes$ of Borel-definable measures is not associative in general [Conant, Gannon, Hanson'21];
  - Not all groups in simple theories are definably amenable [C., Hrushovski, Kruckman, Krupinski, Moconja, Pillay and Ramsey'21].
- Some positive results:
  - A weak generalization of ε-nets for *n*-dependen theories [C.,Towsner'20]
  - $NSOP_1$ is preserved under randomizations [Ben Yaacov, C., Ramsey, 21+]

Independent product of definable types $\otimes$, 1

- Given two global types $p(x), q(y)$, there are usually many different global types $r(x, y)$ satisfying $r(x, y) \supseteq p(x) \cup q(y)$ (as $\mathcal{L}_x(\mathbb{M}) \times \mathcal{L}_y(\mathbb{M}) \subsetneq \mathcal{L}_{xy}(\mathbb{M})$).
- Under additional assumptions on p, there is often a canonical "generic" choice of r not introducing any dependencies between x and y (e.g. not containing $x = y$). We restrict to definable types for simplicity of presentation (but works for invariant types as well).
- Given $A \subseteq B \subseteq \mathbb{M}$, a type $p \in S_x(B)$ is *definable over A* if for every formula $\varphi(x, y) \in \mathcal{L}_{xy}$ there exists a formula $d_p\varphi(y) \in \mathcal{L}_y(A)$ such that

$$\forall b \in B^y,\ \varphi(x, b) \in p \iff \models d_p\varphi(b).$$

- A global type is *definable* if it is definable over some small model.
- A theory is stable if and only if all types are definable [Shelah].

Independent product of definable types $\otimes$, 2

- Assume that $p \in S_x(\mathbb{M}), q \in S_y(\mathbb{M})$ and p is definable. Then $p \otimes q \in S_{xy}(\mathbb{M})$ is defined via

$$\varphi(x, y) \in p \otimes q \iff d_p\varphi(y) \in q$$

  for every $\varphi(x, y) \in \mathcal{L}_{xy}$.
- Equivalently, $p \otimes q = tp(a, b/\mathbb{M})$ for some/any $b \models q$ and $a \models p'|_{\mathbb{M}b}$ (in some $\mathbb{M}' \succ \mathbb{M}$; where $p' \in S_x(\mathbb{M}')$ is the extension of p given by the same definition schema).
- E.g. if p is the non-realized type in $\mathrm{Th}(\mathbb{N}, =)$, then $p(x) \otimes p(y) = p(y) \otimes p(x)$ is axiomatized by

$$\{x \neq a, y \neq a : a \in \mathbb{M}\} \cup \{x \neq y\}.$$

- Assume $p(x) = \{x > a : a \in \mathbb{M}\}$ in $\mathrm{Th}(\mathbb{Q}, <)$. Then

$$p(x) \otimes q(y) = \{x > a, y > a : a \in \mathbb{M}\} \cup \{x > y\} \neq p(y) \otimes q(x).$$

- Hence $\otimes$ is associative, but not commutative (unless T is stable).

Convolution product $*$ of definable types

- Assume now that T expands a group, i.e. there exists a definable functions $\cdot$ such that for some/any $\mathcal{M} \models T$, $(\mathcal{M}_x, \cdot)$ is a group.
- In this case, given definable $p, q \in S_x(\mathbb{M})$, we have a definable type $p * q \in S_x(\mathbb{M})$ via

$$\varphi(x) \in p * q \iff \varphi(x \cdot y) \in p(x) \otimes q(y)$$

  for every $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$.
- Equivalently, $p * q = \mathrm{tp}(a \cdot b/\mathbb{M})$ for some/any $(a, b) \models p \otimes q$ in a larger monster model.
- Let $S_x^{\mathrm{def}}(\mathbb{M})$ be the set of all definable global types. Then $(S_x^{\mathrm{def}}(\mathbb{M}), *)$ is a left-continuous semigroup.
- "Left continuous" means: the map $- * q : S_x^{\mathrm{def}}(\mathbb{M}) \to S_x^{\mathrm{def}}(\mathbb{M})$ is continuous for every fixed $q \in S_x^{\mathrm{def}}(\mathbb{M})$.

Idempotent types

- A type $p \in S_x^{\mathrm{def}}(\mathbb{M})$ is *idempotent* if $p * p = p$.
- E.g. let $\mathcal{M}$ be $(\mathbb{Z}, +, P_{n,\alpha})$, with $(P_{n,\alpha} : \alpha < 2^{\aleph_0})$ naming all subsets of $\mathbb{Z}^n$, for all n.
  Then all types over $\mathcal{M}$ are trivially definable, and idempotent types are precisely the idempotent ultrafilters in the sense of Galvin–Glazer's proof of Hindman's theorem (for every finite partition of $\mathbb{Z}$, some part contains all finite sums of elements of an infinite set), see e.g. [Andrews, Goldbring'18].
- In stable theories, idempotent types are known to arise from type-definable subgroups (group chunk theorem and its variants [Hrushovski, Newelski]).
- This is parallel to the following classical line of research:

Motivation: analogy with the classical (locally-)compact case

- Let G be a locally compact topological group.
- Then the space of regular Borel probability measures on G is equipped with the *convolution product*:

$$\mu * \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y) d\mu(x) d\nu(y)$$

  for a Borel set $A \subseteq G$.
- If G is compact, then μ is idempotent if and only if the support of μ is a compact subgroup of G and μ restricted to it is the (bi-invariant) Haar measure [Wendel'54].
- Same characterization extends to locally compact abelian groups [Rudin'59, Cohen'60].
- Compact (semi-)topological semigroup — the picture becomes more complicated [Glicksber'59, Pym'69, ...].

Independent product $\otimes$ of definable Keisler measures

- We would like to find a parallel for Keisler measures, generalizing the situation for types. First, need to make sense of the convolution product.
- A Keisler measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ is *definable* (over $\mathcal{M} \preceq \mathbb{M}$) if:
  1. for any $\varphi(x, y) \in \mathcal{L}_{xy}$ and $b \in \mathbb{M}_y$, $\mu(\varphi(x, b))$ depends only on $\mathrm{tp}(b/\mathcal{M})$
     (in which case, given $q \in S_y(\mathcal{M})$, we write $\mu(\varphi(x, q))$ to denote $\mu(\varphi(x, b))$ for some/any $b \models q$);
  2. the map $q \in S_y(\mathcal{M}) \mapsto \mu(\varphi(x, q)) \in [0, 1]$ is continuous.
- A type $p \in S_x(\mathbb{M})$ is definable as a type iff it is definable as a measure.
- Given $\mu \in \mathfrak{M}_x(\mathbb{M}), \nu \in \mathfrak{M}_y(\mathbb{M})$ with μ $\mathcal{M}$-definable, we can define $\mu \otimes \nu \in \mathfrak{M}_{xy}(\mathbb{M})$ via

$$\mu \otimes \nu(\varphi(x, y)) := \int_{S_y(\mathcal{M})} \mu(\varphi(x, q)) d\nu|_{\mathcal{M}}(q).$$

- The integral makes sense by (2), viewing $\nu|_{\mathcal{M}}$ as a regular Borel measure on $S_y(\mathcal{M})$. (Works also for only *Borel-definable*).

Convolution product $*$ of definable Keisler measures

- $\otimes$ on definable measures extends $\otimes$ on definable types defined earlier.
- If now T expands a group, given definable $\mu, \nu \in \mathfrak{M}_x(\mathbb{M})$, we get a definable $\mu * \nu \in \mathfrak{M}_x(\mathbb{M})$ via

$$\mu * \nu(\varphi(x)) := \mu_x \otimes \nu_y(\varphi(x \cdot y)).$$

- Again, restricting to definable types, we recover $*$ defined earlier.
- The set of all definable Keisler measures with $*$ is a semigroup. A measure μ is idempotent if $\mu * \mu = \mu$.

Theorem (C., Gannon'20)

If T is NIP, then $*$ is again left-continuous (on invariant measures).

- In general T — unclear.

Idempotent Keisler measures vs the classical locally compact case

- First of all, in general a definable group has no non-discrete topology.
- Given $\mu \in \mathfrak{M}_x(A)$, its *support* is

$$S(\mu) := \{p \in S_x(A) : \varphi(x) \in p \implies \mu(\varphi(x)) > 0\}.$$

  It is a closed non-empty subset of $S_x(A)$.
- As we mentioned, in a locally compact topological group, support of an idempotent measure is a closed subgroup — no longer true for idempotent Keisler measures (with respect to $*$ on types), even if there is some nice topology present:

Supports of idempotent Keisler measures: an example, 1

- Let $\mathcal{M} = (S^1, \cdot, C(x, y, z))$ be the compact unit circle group (of rotations) over $\mathbb{R}$, with C the cyclic clockwise ordering.
- Let $\mu \in \mathfrak{M}_x(\mathbb{M})$ be given by $\mu(\varphi(x)) = h(\varphi(\mathcal{M}))$ for $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, where h is the Haar measure on S^1.
- Then μ is definable and right translation invariant (by elements of $\mathbb{M}$), hence idempotent.
- Let $\mathrm{st} : S_x(\mathbb{M}) \to \mathcal{M}$ be the standard part map. Assume that $p \in S(\mu)$ and $\mathrm{st}(p) = a$. Then $\varphi_\varepsilon(x) := C(a - \varepsilon, x, a + \varepsilon) \notin p$ for every infinitesimal $\varepsilon \in \mathbb{M}$ ($x \neq a \in p$ as $h(x = a) = 0$, and if $\varphi_\varepsilon(x) \in p$, then $\mu(\varphi_\varepsilon(x) \wedge x \neq a) > 0$, but $\varphi_\varepsilon(\mathcal{M}) = \{a\}$ — a contradiction).
- As the types in $S_x(\mathbb{M})$ are determined by the cuts in the circular order, it follows that for every $a \in \mathcal{M}$ there are exactly two types $a_+(x), a_-(x) \in S(\mu)$ determined by whether $C(a + \varepsilon, x, b)$ holds for every infinitesimal ε and $b \in \mathcal{M}$, or $C(b, x, a - \varepsilon)$ holds for every infinitesimal ε and $b \in \mathcal{M}$, respectively.

Supports of idempotent Keisler measures: an example, 2

- It follows that $(S(\mu), *) \cong S^1 \times \{+, -\}$ with multiplication defined by:
$$a_\delta * b_\gamma = (a \cdot b)_\delta$$
for all $a, b \in S^1$ and $\delta, \gamma \in \{+, -\}$.
- Hence $(S(\mu), *)$ is not a group (as it has two idempotents).
- This group is NIP (definable in an *o*-minimal theory), unstable.

Supports of idempotent Keisler measures: a theorem

- Adapting Glicksberg, we show:

Theorem (C., Gannon'20)

1. *(T arbitrary) Let $\mu \in \mathfrak{M}_x(\mathbb{M})$ be an idempotent definable and invariantly supported Keisler measure. Then $(S(\mu), *)$ is a compact, left continuous semigroup with no closed two-sided ideals.*
2. *(T NIP) The same conclusion holds just assuming that μ is definable.*

- Where:
  - $I \subseteq S(\mu)$ is a left (right) ideal if: $q \in I \implies p * q \in I$ (resp., $q * p \in I$) for every $p \in S(\mu)$. Two-sided = both left and right.
  - μ is *invariantly supported* if there exists a small model $\mathcal{M} \preceq \mathbb{M}$ s.t. every $p \in S(\mu)$ is $\mathrm{Aut}(\mathbb{M}/\mathcal{M})$-invariant.

Type-definable subgroups

- Instead of closed subgroups in the topological setting, we consider *type-definable* subgroups.
- Assume that $\mathbb{M} \models T$ expands a group, and $\mathcal{H}$ is a type-definable subgroup of $(\mathbb{M}, \cdot)$ (i.e. the underlying set of $\mathcal{H}$ can be defined by a small partial type $H(x)$ with parameters in $\mathbb{M}$).
- Let $\mathcal{H}$ be type-definable and suppose that $\mu \in \mathfrak{M}_x(\mathbb{M})$ is concentrated on $\mathcal{H}$ (i.e. $p \in S(\mu) \implies p(x) \vdash H(x)$) and is right $\mathcal{H}$-invariant (i.e. for any $\varphi(x) \in \mathcal{L}_x(\mathbb{M}), a \in \mathcal{H}$, $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$). Then μ is idempotent.
- Ideology: by analogy with the classical case, we expect all idempotent Keisler measures in model-theoretically tame groups to be of this form.
- (Translation-invariant Keisler measures in NIP groups are classified: the ergodic ones are described as certain liftings of the Haar measure on the canonical compact quotient G/G^{00} [C., Simon'18].)

Idempotent measures in stable groups

- Can confirm for stable groups:

Theorem (C., Gannon'20)

Let T be a stable theory expanding a group and $\mu \in \mathfrak{M}_x(\mathbb{M})$ a Keisler measure. TFAE:

1. *μ is idempotent;*
2. *μ is the unique right/left-invariant measure on its stabilizer, i.e. the type-definable subgroup $St(\mu) = \{g \in \mathbb{M} : g \cdot \mu = \mu\}$.*

- The following groups are stable: abelian, free, algebraic over $\mathbb{C}$ (e.g. $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$, abelian varieties).
- Ingredients: structure of the supports of definable idempotent measures in NIP; definability of all measures in stable theories (and type-definability of their stabilizers); a variant of Hrushovski's group chunk theorem for partial types due to Newelski.

Idempotent measures in NIP

- Can we classify idempotent measures in NIP, or even more generally?
- Conjecture: in a (definably amenable) NIP group, every idempotent definable (invariant) measure μ is a left-invariant measure on its type-definable (invariant) stabilizer subgroup.
- Note: no longer needs to be unique!
- Work in progress: can confirm under some additional assumptions: abelian group, μ generically stable (in which case it is the unique measure on its type-definable stabilizer).