Distality in Valued Fields and Related Structures

joint w/ Aschenbrenner, Gehret and Ziegler

Main Theorem

Let K be a $\overset{\text{inf.}}{\text{hens.}}$ val. field, viewed as $(K, \mathcal{O})$, with value grp. Γ and residue field k. Then K is distal (resp., has a distal exp) iff:

1) K is char $(0,0)$ or K is char. 0, k is finite, and $\mathcal{O}$ is finitely ramified (i.e. $[0, v(n)]$ is finite for all $n \geq 1$).

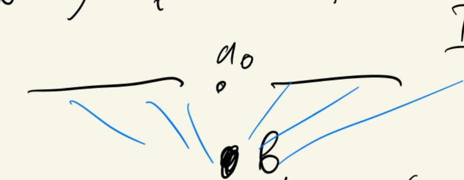
2) both k and Γ are distal (resp, have distal expansions).

- "Shelah's conjecture".
- [Anscombe, Jahnke]

Distality

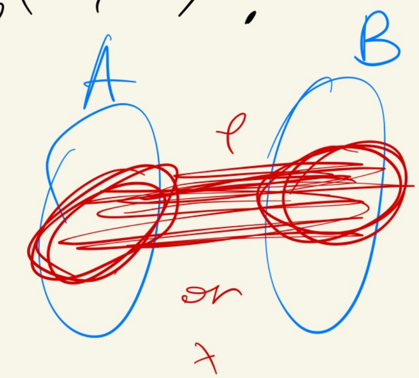
<u>Fact</u> T complete, $\mathbb{M} \models T$. TFAE:

1) For any indisc. seq. $I = (a_i : i \in \mathbb{Q})$ and b, if $(a_i : i \neq 0)$ is indisc / b, then I is indisc. / b.



2) T has "~~strong~~ honest defs". For every $\varphi(x,y)$ there exists $\psi(x;y_1,\dots,y_n)$ s.t. $\forall B \subseteq_{fin} \mathbb{M}_y$, $|B| \geq 2$, $a \in \mathbb{M}_x$, $\exists b_1,\dots,b_n \in B$ s.t. $\models \psi(a;b_1,\dots,b_n)$ and $\psi(x;b_1,\dots,b_n) \vdash tp_\varphi(a/B)$.

3) Definable strong Erdős-Hajnal property: $\forall \varphi(x,y)$, $\exists \varepsilon > 0$, $\psi_1(x,z_1)$, $\psi_2(y,z_2)$ s.t. $\forall A \subseteq_{fin} \mathbb{M}_x$, $B \subseteq_{fin} \mathbb{M}_y$, $\exists c_i \in \mathbb{M}_{z_i}$ s.t. $|\psi_1(A,c_1)| \geq \varepsilon |A|$, $|\psi_2(B,c_2)| \geq \varepsilon |B|$, and either $\psi_1(A,c_1) \times \psi_2(B,c_2) \subseteq \varphi(A,B)$ or $\subseteq A \times B \setminus \varphi(A,B)$.



- 1) is def. of distality [Simon] $\cancel{\Leftarrow}$ 1) $\Rightarrow$ NIP.

  1) $\Leftrightarrow$ 2)$_{(weak)}$ [C., Simon], $\Leftrightarrow$ 3) [C, Starchenko].

- <u>Ex</u>: o-min, p-adics, ...

  - In 1), wlog a_i or b is a singleton.

Prop Let T be NIP, D is $\emptyset$-definable with D_{ind} is distal $((D, R, \dots)$, $R = D^n \cap E$, E $\emptyset$-def$)$. If

① For any $b \in \mathbb{M}$ and $I = (a_i : i \in \mathbb{Q})$ in D, if $(a_i : i \neq 0)$ is indisc$/b$, then I is indisc$/b$.

② For any $I = (a_i : i \in \mathbb{Q})$ in $\mathbb{M}$ and $b \in D$, if $(a_i : i \neq 0)$ is indisc$/b$, then I is indisc$/b$.

Cor 1 If $\mathbb{M}$ is NIP, D is $\emptyset$-def. with D_{ind} is distal, $\mathbb{M} \subseteq \mathrm{acl}(D)$, then $\mathbb{M}$ is distal.

Cor 2 T is distal $\iff T^{eq}$ is distal.

Distal fields and rings

Fact [C, Starchenko] If K is an inf. field def. in a distal T, then $\mathrm{char}(K) = 0$.

Reason the $\mathbb{F}$-def. point-line incidence rel. in $\mathbb{F}_p^{alg}$ doesn't satisfy sEH.

[Kaplan, Scanlon, Wagner] K is NIP, char $p \Rightarrow \mathbb{F}_p^{alg} \subseteq K$.

Prob. K is type-def. in a distal str. $\Rightarrow$ K char 0?

Prop Supp. R is a unital ring w/o zero-divisors def. in a distal T. Then $\mathrm{char}(R) = 0$ (~the smallest $n \geq 1$ s.t. $n \cdot 1 = 0$, or 0 o/w).

Prop "no zero-div" can't be dropped.

$H = \mathbb{F}_p^{\mathbb{Q}}$ has a distal exp (naming val)

$R := \mathbb{F}_p \times H$ — comm. ring, of char p.

- Work in progr. w/ Simon: all ab. grps have dist exps.

Fact: If $(K, \mathcal{O})$ is NIP and k is finite, then v is finitely ramified.

So: If $(K,\mathcal{O})$ has a dist. exp., then $(K,\mathcal{O})$ is fin. ram, and k has char 0 or is finite.

Reduction to RV

$(K,\mathcal{O})$, char $K=0$. For $\delta\in\Gamma^{\geq 0}$, let $\mathfrak{m}_\delta=\{x\in K : v(x)>\delta\}$ - an ideal of $\mathcal{O}$.

$\mathfrak{m}=\mathfrak{m}_0$.

$RV_\delta := K/(1+\mathfrak{m}_\delta)$, $rv_\delta : K\to RV_\delta$

$RV=RV_0$.

SES of ab. grps: $1\to k^\times \to RV^\times \xrightarrow{v_{rv}} \Gamma \to 0$

on RV_δ (as $a\in\mathcal{O}\setminus\mathfrak{m}$, $a(1+\mathfrak{m})\in RV^\times$ only dep. on "$a+\mathfrak{m}$").

$\oplus_\delta(r,s,t) \iff \exists x,y,z\in K\ (r=rv_\delta(x)\wedge s=rv_\delta(y)\wedge t=rv_\delta(z)\wedge x+y=z)$.

$$\boxed{\overline{RV}} = \begin{cases} (RV_0,\oplus_0,\times) & \text{if char } k=0\\ (RV_\delta,\oplus_\delta,\times, rv_{\gamma\to\delta})_{\gamma\geq\delta\in\{v(p^n)\}_{n\in\mathbb{N}}} & \text{if char } k=p.\end{cases}$$

Note: $\overline{RV}$ is interp. in $(K,\mathcal{O})$.

Fact [Flenner] $(K,\mathcal{O})$ has QE down to $\overline{RV}$

1) char$(K)=0$, hens $\Rightarrow$ $(K,\mathcal{O})$ has QE down to $\overline{RV}$

2) $\overline{RV}$ is fully stably embedded.

Rem holds "resplendently".

Using analysis of indisc. seq,

Prop $(K, \mathcal{O})$ is distal (has dist exp) Iff it is fin. ram.
and $\overline{RV}$ is dist (has dist exp).

Reduction to k and Γ

Prop. $\overline{RV}$ is distal Iff k and Γ are.
(has a dist exp)

char $k > 0$ — treated by the "finite covers preserve dist" lemma, $\to RV_{v(p^2)} \to RV_{v(p)}$.

char 0: RV is a reduct of $\boxed{k \times \Gamma}$
$\overset{||}{RV_0}$ k finite

Let $0 \to \underbrace{A}_{} \xrightarrow{i} B \xrightarrow{\nu} \underbrace{C}_{} \to 0$ — $\boxed{\text{pure}}$ SES of ab. grps.

$\swarrow_{\pi'}$

~ 3-sorted struct.

‖

$i(A)$ is a pure subgrp. of B,

(if $a \in i(A)$ and $nx = a$ has a sol. in B, then has a sol. in $i(A)$).

Add sorts:

- A/nA $\quad \forall n$

$\pi_n : A \to A/nA$

$p_n : B \to A/nA$ on $\nu^{-1}(nC)$ — the comp. of group. morph.

$\nu^{-1}(nC) = nB + i(A) \to (nB + i(A))/nB \xrightarrow{\sim} i(A)/(nB \cap i(A))$

$\xrightarrow{\sim} A/nA$.

outside of $\nu^{-1}(nC)$ — $= 0$.

Note: $\pi_n = p_n \circ i$

if SES splits, $p_n = \pi_n \circ \pi'$ on $\nu^{-1}(nC)$

L_{AC} — the group. lang on A and on C, with arb. add struct.

L_B — ~~fo~~ the grp. lang on B

<u>Thm</u> Modulo the theory of pure SES, every $\varphi(\overbrace{x_A, x_B, x_C}^{\text{tuples}})$ is equiv. $\varphi'(x_A, \sigma_1(x_B), \dots, \sigma_m(x_B), x_C)$ where $\varphi' \in L_{AC} \cup \{\pi_0, \pi_1, \dots\}$ and $\sigma_i(x_B) = \begin{cases} p_n(t(x_B)) \\ \nu(t(x_B)) \end{cases}$, where t is an L_B-term.

p-istality in OAG's

(in pure ord. grp. lang).

<u>Thm</u> Let $(G, +, <)$ be a str. dep. OAG,

G is distal $\iff$ G is dp-~~min~~ min $\iff$ $|G/pG| < \infty$ $\forall$ prime p.

<u>Conj</u> All OAG's have distal exp's.

"Shelah's conj" + $\Rightarrow$ An NIP field K ~~is distal~~ has a dist. exp

iff

K has a hens. val. with res. field ACF_0, RCF or finite

iff

K doesn't interp. an inf. field of positive char.

Ramanujan graph. [Nesetril, Ossona Mendez, ...]

[~~Kest~~ Boxall, Kestner] M is distal $\iff$ M^{sh} is distal.