Combinatorial properties of generically stable measures

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Szemerédi regularity lemma

Theorem

[E. Szemerédi, 1975] Every large enough graph can be partitioned into boundedly many sets so that on almost all pairs of those sets the edges are approximately uniformly distributed at random.

Szemerédi regularity lemma

Theorem

[E. Szemerédi, 1975] Given $\varepsilon > 0$, there exists $K = K(\varepsilon)$ such that: for any finite bipartite graph $R \subseteq A \times B$, there exist partitions $A = A_1 \cup \ldots \cup A_k$ and $B = B_1 \cup \ldots \cup B_k$ into non-empty sets, and a set $\Sigma \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$ of good pairs with the following properties.

1. *(Bounded size of the partition)* $k \leq K$.
2. *(Few exceptions)* $\left| \bigcup_{(i,j) \in \Sigma} A_i \times B_j \right| \geq (1 - \varepsilon) |A| |B|$.
3. *(ε-regularity) For all $(i,j) \in \Sigma$, and all $A' \subseteq A_i, B' \subseteq B_j$,*

$$\left| \left| R \cap \left(A' \times B' \right) \right| - d_{ij} \left| A' \right| \left| B' \right| \right| \leq \varepsilon |A| |B|,$$

where $d_{ij} = \frac{\left| R \cap \left(A_i \times B_j \right) \right|}{\left| A_i \times B_j \right|}$.

Szemerédi regularity lemma: bounds and applications

- Exist various versions for weaker and stronger partitions, for hypergraphs, etc.
- Increasing the error a little one may assume that the sets in the partition are of (approximately) equal size.
- Has many applications in extreme graph combinatorics, additive number theory, computer science, etc.
- [T. Gowers, 1997] The size of the partition $K(\varepsilon)$ grows as an exponential tower $2^{2^{\cdots}}$ of height $\left(1/\varepsilon^{\frac{1}{64}}\right)$.
- For restricted families of graphs (e.g. coming from algebra, geometry, etc.) one can obtain stronger regularity. Some recent positive results fit nicely into the *model-theoretic* classification picture.

Model-theoretic setting, 1

- Let $\mathcal{M} = (M, R_i, f_i, c_i)$ denote a first-order structure with some distinguished relations $R_i \subseteq M^{k_i}$, functions $f_i : M^{k_i} \to M$ and constants $c_i \in M$.
- A (partitioned) first-order formula $\phi(x, y)$ is an expression of the form $\forall z_1 \exists z_2 \ldots \forall z_{2n-1} \exists z_{2n} \psi(x, y, \bar{z})$, where ψ is a Boolean combination of the (superpositions of) basic relations and functions, and x, y are tuples of variables. Let L be the collection of all formulas.
- Given $b \in M_y$, $\phi(x, b)$ is *an instance* of ϕ and defines a set $\phi(M, b) = \{a \in M_x : \mathcal{M} \models \phi(a, b)\}$.
- For every formula $\phi(x, y) \in L$, we have a *definable family* $\mathcal{F}_\phi = \{\phi(M, b) : b \in M_y\}$ of subsets of M_x.
- Let $\mathrm{Def}(M_x) \subseteq \mathcal{P}(M_x)$ be the Boolean algebra of all definable subsets.
- Let $\mathrm{Def}_\phi(M_x) \subseteq \mathcal{P}(M_x)$ be the Boolean algebra of all subsets defined by Boolean combinations of instances of ϕ.

Model-theoretic setting, 2

- $\mathcal{M} = (\mathbb{C}, +, \times, 0, 1)$. By quantifier elimination, definable subsets of M_x are the constructible ones (i.e. Boolean combinations of polynomial equalities).
- $\mathcal{M} = (\mathbb{R}, +, \times, <, 0, 1)$. By quantifier elimination, definable subsets of M_x are the semialgebraic ones (Boolean combinations of polynomial equalities and inequalities).
- $\mathcal{M} = (\mathbb{Q}_p, +, \times, 0, 1)$. By Macintyre, eliminates quantifiers after adding $v(x) < v(y)$ and $P_n(x) \iff \exists z (x = z^n)$ for $n \geq 2$.
- In $\mathbb{C}$ or $\mathbb{R}$, given $\phi \in L$, all sets in the definable family $\mathcal{F}_\phi$ have description complexity $\leq d = d(\phi)$. And conversely, the family of all (semi-)algebraic subsets of M_x of the description complexity $\leq d$ is of the form $\mathcal{F}_\phi$ for some $\phi \in L$.

Model-theoretic setting, 3

- A *Keisler measure* μ is a finitely additive probability measure on $\mathrm{Def}\,(M_x)$.

Example.

- Complete types over M correspond to zero-one Keisler measures.
- Given a finite set $A \subseteq M_x$, taking $\mu\,(X) = \frac{|X \cap A|}{|A|}$ for every $X \in \mathrm{Def}\,(M_x)$ defines a Keisler measure ("counting measure").
- Let λ_n be the Lebesgue measure on the unit cube $[0,1]^n$ in $\mathbb{R}^n$. By QE, if $X \in \mathrm{Def}\,(\mathbb{R}^n)$ then $X \cap [0,1]^n$ is λ_n-measurable, hence λ_n induces a Keisler measure.
- Let λ be the (normalized) Haar measure in $\mathbb{Q}_p$ restricted to a compact ball. Again, by QE all definable sets are λ-measurable and λ induces a Keisler measure.

Shelah's classification

- Motivated by Morley's conjecture (counting the number of uncountable models of first-order theories), Shelah has introduced a number of "dividing lines" which can be expressed as measuring the combinatorial complexity of the definable families $\mathcal{F}_\phi$.
- Classification picture: see e.g. http://www.forkinganddividing.com/.
- A lot of tools were developed for analyzing types in stable theories (Shelah, Zilber, Hrushovski and many others).
- More recently, generalizing these tools to measures in NIP has attracted a lot of attention.

Generically stable measures

- We concentrate on *generically stable* Keisler measures, which play a particularly important role in NIP theories (Keisler, Shelah, Peterzil, Pillay, Hrushovski, Simon).

Definition. A Keisler measure μ on M_x is *generically stable* if for every formula $\phi(x, y) \in L$ and $\varepsilon > 0$ there are some $a_1, \ldots, a_m \in M_x$ (possibly with repetitions) such that

$$\left| \mu(\phi(x, b)) - \frac{|\{i : a_i \in \phi(M, b)\}|}{m} \right| < \varepsilon$$

for every $b \in M_y$.

- In other words, the *VC-theorem* holds for μ.
- Counting, Lebesgue and Haar measures are all generically stable.
- The type at $+\infty$ in $(\mathbb{R}, +, \times, <, 0, 1)$ is not generically stable.

Product measures, 1

- Assume we are given a definable relation $E(x, y) \in \mathrm{Def}(M_{xy})$.
- Let μ and ν be Keisler measures on M_x and M_y, respectively.
- Note that $\mathrm{Def}(M_{xy}) \neq \mathrm{Def}(M_x) \times \mathrm{Def}(M_y)$, and E may not be $\mu \times \nu$-measurable.
- In general, there are many ways to extend the product measure $\mu \times \nu$ to a measure ω on $\mathrm{Def}(M_{xy})$.
- For generically stable measures, we have a canonical choice.

Definition. Given generically stable measures μ, ν, on M_x, M_y respectively, we define a measure $\mu \otimes \nu$ on M_{xy} by

$$\mu \otimes \nu (E(x, y)) = \int_{M_x} \left(\int_{M_y} \mathbf{1}_E(x, y)\, d\mu_y \right) d\mu_x$$

- By generic stability, it is well-defined and $\mu \otimes \nu = \nu \otimes \mu$.

Product measures, 2

- If μ is the counting measure on A, then $\mu \otimes \mu$ is the counting measure on $A \times A$.
- If λ is the Lebesgue measure on $[0, 1]$, then $\lambda \otimes \lambda$ is the Lebesgue measure on $[0, 1]^2$.
- Etc.

Stable case, 1

Definition.

1. A formula $\phi(x, y)$ is k-stable if **there are no** $(a_i : i < k)$ in M_x and $(b_i : i < k)$ in M_y such that

$$M \models \phi(a_i, b_j) \iff i \leq j.$$

2. A formula is *stable* if it is k-stable for some $k \in \omega$.
3. M is stable if all formulas are stable.

Examples of stable structures.

1. Abelian groups and modules,
2. $(\mathbb{C}, +, \times, 0, 1)$,
3. [Sela] free groups (in the pure group language $(\cdot, ^{-1}, 0)$),
4. Planar graphs (in the language with a single binary relation).

Stable case, 2

- If M is stable, then all Keisler measures are generically stable (follows from classical stability theory).

Theorem

[Malliaris, Shelah, 2014], [Malliaris, Pillay, 2016] Let M be stable. For every definable $E(x_1, \ldots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any Keisler measures μ_i on M_{x_i} there are partitions $M_{x_i} = \bigcup_{j<K} A_{i,j}$ satisfying

1. $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
2. *for all* $(i_1, \ldots, i_n) \in \{1, \ldots, K\}^n$ *and definable* $A'_1 \subseteq A_{1,i_1}, \ldots, A'_n \subseteq A_{n,i_n}$ *either* $d_E(A'_1, \ldots, A'_n) < \varepsilon$ *or* $d_E(A'_1, \ldots, A'_n) > 1 - \varepsilon$.
3. *Each* $A_{i,j}$ *is defined by an instance of an* E*-formula depending only on* E *and* ε.

Here the density $d_E(A'_1, \ldots, A'_n) = \frac{\mu(E \cap A'_1 \times \ldots \times A'_n)}{\mu(A'_1 \times \ldots \times A'_n)}$, *where* $\mu = \mu_1 \otimes \ldots \otimes \mu_n$.

Stable case, 3

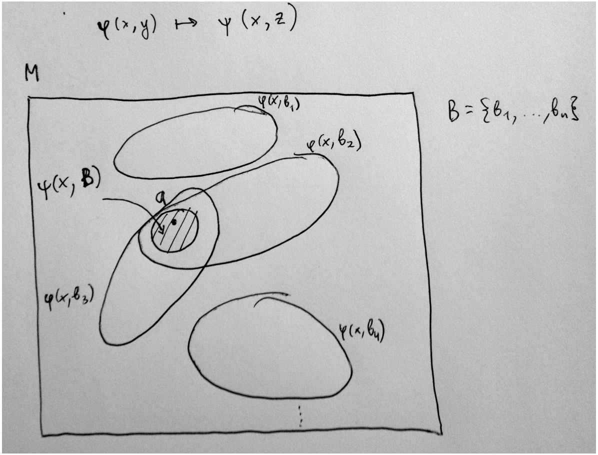
- [Malliaris, Shelah] for finite counting measures, the polynomial bound is explicit.
- [Malliaris, Pillay] for general measures, but polynomial bound is not discussed.
- Can be combined, and generalized to hypergraphs.
- Only assumes local stability, so applies to the family of all finite k-stable graphs (by taking an ultraproduct of counterexamples).

Distal case, 1

- The class of *distal theories* was introduced and studied by [Simon, 2011] in order to capture the class of "purely unstable" NIP structures.
- The original definition is in terms of a certain property of indiscernible sequences.
- [C., Simon, 2012] give a combinatorial characterization of distality (our proof uses the (p, q)-theorem of Alon-Kleitman-Matousek for families of finite VC-dimension):

Distal structures

- **Theorem/Definition** An NIP structure M is *distal* if and only if for every definable family $\{\phi(x,b) : b \in M^d\}$ of subsets of M there is a definable family $\{\psi(x,c) : c \in M^{kd}\}$ such that for every $a \in M$ and every finite set $B \subset M^d$ there is some $c \in B^k$ such that $a \in \psi(x,c)$ and for every $a' \in \psi(x,c)$ we have $a' \in \phi(x,b) \Leftrightarrow a \in \phi(x,b)$, for all $b \in B$.



Examples of distal structures

- All (weakly) *o*-minimal structures, e.g. $M = (\mathbb{R}, +, \times, e^x)$.
- Presburger arithmetic.
- Any *p*-minimal theory with Skolem functions is distal. E.g. $(\mathbb{Q}_p, +, \times)$ for each prime p is distal (e.g. due to the *p*-adic cell decomposition of Denef).
- The (valued differential) field of transseries.

Distal regularity lemma

Theorem

[C., Starchenko] Let M be distal. For every definable $E(x_1, \ldots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any generically stable Keisler measures μ_i on M_{x_i} there are partitions $M_i = \bigcup_{j<K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \ldots, K\}^n$ such that

1. $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
2. $\mu\left(\bigcup_{(i_1,\ldots,i_n)\in\Sigma} A_{1,i_1} \times \ldots \times A_{n,i_n}\right) \geq 1 - \varepsilon$, *where* $\mu = \mu_1 \otimes \ldots \otimes \mu_n$.
3. *for all* $(i_1, \ldots, i_n) \in \Sigma$, *either* $(A_{1,i_1} \times \ldots \times A_{n,i_n}) \cap E = \emptyset$ *or* $A_{1,i_1} \times \ldots \times A_{n,i_n} \subseteq E$.
4. *Each* $A_{i,j}$ *is defined by an instance of a formula* $\psi_i(x_i, z)$ *which only depends on* E *(and not on* ε*!).*

Semialgebraic case

- Generalizes the very important semialgebraic case due to [Fox, Gromov, Lafforgue, Naor, Pach, 2012] and [Fox, Pach, Suk, 2015].
- But also applies to graphs definable in the p-adics, with respect to the Haar measure.
- Many questions about the optimality of the bounds remain, in the o-minimal and the p-adic cases in particular.

General NIP case

Definition.

1. A formula $\phi(x, y)$ is NIP if the family $\mathcal{F}_\phi$ has finite VC-dimension.
2. M is NIP if all definable formulas are NIP.

- The class of NIP was introduced by Shelah around the same time as VC theory was being developed.
- Attracted a lot of attention recently in model theory (important algebraic examples such as ACVF, as well as generalizing methods of stability).
- [Lovasz, Szegedy, 2010] prove a strong regularity lemma for graphs (and graphons) of finite VC dimension.
- We give a model-theoretic version of this result, generalizing the stable and the distal cases.

General NIP case, 2

Theorem

[C., Starchenko] Let M be NIP. For every definable $E(x_1, \ldots, x_n)$ there is some $c = c(E)$ such that: for any $\varepsilon > 0$ and any generically stable Keisler measures μ_i on M_{x_i} there are partitions $M_i = \bigcup_{j<K} A_{i,j}$ and a set $\Sigma \subseteq \{1, \ldots, K\}^n$ such that:

1. $K \leq \left(\frac{1}{\varepsilon}\right)^c$.
2. $\mu\left(\bigcup_{(i_1,\ldots,i_n)\in\Sigma} A_{1,i_1} \times \ldots \times A_{n,i_n}\right) \geq 1 - \varepsilon$, *where* $\mu = \mu_1 \otimes \ldots \otimes \mu_n$,
3. *for all* $(i_1, \ldots, i_n) \in \Sigma$ *and definable* $A'_1 \subseteq A_{1,i_1}, \ldots, A'_n \subseteq A_{n,i_n}$ *either* $d_E(A'_1, \ldots, A'_n) < \varepsilon$ *or* $d_E(A'_1, \ldots, A'_n) > 1 - \varepsilon$.
4. *each $A_{i,j}$ is defined by an instance of an E-formula depending only on E and ε.*

General NIP case, 3

- The proof relies on the theory of integration for finitely additive measures, and some basic theory of generically stable measures, along with the efficient packing lemma from Lovasz-Szegedy.
- This covers the case of finite graphs of finite VC-dimension:
  - Let $(M_i)_{i \in \omega}$ be a sequence of finite L-structures, let $M = \prod M_i / \mathcal{U}$ for $\mathcal{U}$ a non-principal ultrafilter. For an L-definable subset $X \subseteq M$, let $\mu(X) = \lim_{\mathcal{U}} \frac{|X_i|}{|M_i|}$ be the non-standard counting measure on M. By the VC-theorem for finite counting measures, μ is generically stable. Then the theorem applies to the ultraproduct of counterexamples.

Finding large homogeneous subsets

- These results are related to the question of finding a "large" (approximately) homogeneous subset in a definable (hyper-)graph.
- E.g. Erdős-Hajnal conjecture, Rödl's theorem, etc.
- Stable case: everything holds. [Malliaris, Shelah], [C., Starchenko].
- In NIP — not so clear.

Rödl property

- Let M be a structure and let $\mathcal{M}$ be a class of Keisler measures. Let $\mathcal{E}$ be a collection of definable (symmetric) (hyper-)graphs in some powers of M.
- We will say that $\mathcal{E}$ satisfies the *Rödl property* with respect to $\mathcal{M}$ if for every $\varepsilon > 0$ there is some $\delta > 0$ such that for every $E \subseteq (M^n)^k$ in $\mathcal{E}$ and every $\mu \in \mathcal{M}$ a Keisler measure on M^n, there is some definable $A \subseteq M^n, \mu(A) > \delta$ such that the $\mu \otimes \mu$-density of E on A is either $< \varepsilon$ or $> 1 - \varepsilon$.
- Note: bipartite versions are always easier, and follow directly from the corresponding regularity lemmas.

Rödl's theorem

- [Rödl, rephrased] Let M be an ultraproduct of finite graphs $M_i = (V_i, E_i)$ with an NIP edge relation and L the language of set theory. Let μ be the ultraproduct of counting measures. Then $\{E\}$ satisfies the Rödl property for $\{\mu\}$.
- Question: can this be generalized to arbitrary generically stable measures in NIP structures?
- [C., Starchenko] Yes (at least non-uniformly) for the Lebesgue measure in o-minimal theories.
- No (at least not uniformly), for the Haar measure in the p-adics.
- Example: $E(x, y)$ holds if and only if $v(x - y)$ is odd (i.e. if the branches x and y split at an odd level). This is a symmetric relation definable in the Macintyre's language. The density of E on a ball is always bounded away from 0 and 1, and every definable set of positive measure contains a ball of positive measure.