Idempotent Keisler measures (and convolution semigroups)

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Spaces of types

- Let T be a complete first-order theory in a language $\mathcal{L}$, $\mathbb{M} \models T$ a monster model (i.e. κ-saturated and κ-homogeneous for a sufficiently large cardinal κ), $\mathcal{M} \preceq \mathbb{M}$ a small elementary submodel.
- For $A \subseteq \mathbb{M}$ and x an arbitrary tuple of variables, $S_x(A)$ denotes the set of complete types over A.
- Let $\mathcal{L}_x(A)$ denote the set of all formulas $\varphi(x)$ with parameters in A, up to logical equivalence — which we identify with the Boolean algebra of A-definable subsets of $\mathbb{M}_x$; $\mathcal{L}_x := \mathcal{L}_x(\emptyset)$.
- Then the types in $S_x(A)$ are the ultrafilter on $\mathcal{L}_x(A)$.
- By Stone duality, $S_x(A)$ is a totally disconnected compact Hausdorff topological space with a basis of clopen sets of the form
$$\langle \varphi \rangle := \{p \in S_x(A) : \varphi(x) \in p\}$$
for $\varphi(x) \in \mathcal{L}_x(A)$.
- We refer to types in $S_x(\mathbb{M})$ as *global types*.

Keisler measures

- A *Keisler measure* μ in variables x over $A \subseteq \mathbb{M}$ is a finitely-additive probability measure on the Boolean algebra $\mathcal{L}_x(A)$ of A-definable subsets of $\mathbb{M}_x$.
- $\mathfrak{M}_x(A)$ denotes the set of all Keisler measures in x over A.
- Then $\mathfrak{M}_x(A)$ is a compact Hausdorff space with the topology induced from $[0,1]^{\mathcal{L}_x(A)}$ (equipped with the product topology).
- A basis is given by the open sets

$$\bigcap_{i<n} \{\mu \in \mathfrak{M}_x(A) : r_i < \mu(\varphi_i(x)) < s_i\}$$

  with $n \in \mathbb{N}$ and $\varphi_i \in \mathcal{L}_x(A), r_i, s_i \in [0,1]$ for $i < n$.
- Identifying p with the Dirac measure δ_p, $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$ (and the convex hull of $S_x(A)$ is dense).
- Every $\mu \in \mathfrak{M}_x(A)$, viewed as a measure on the clopen subsets of $S_x(A)$, extends uniquely to a regular (countably additive) probability measure on Borel subsets of $S_x(A)$; and the topology above corresponds to the weak*-topology: $\mu_i \to \mu$ if $\int f d\mu_i \to \int f d\mu$ for every continuous $f : S_x(A) \to \mathbb{R}$.

Some examples of Keisler measures

1. In arbitrary T, given $p_i \in S_x(A)$ and $r_i \in \mathbb{R}$ for $i \in \mathbb{N}$ with $\sum_{i\in\mathbb{N}} r_i = 1$, $\mu := \sum_{i\in\mathbb{N}} r_i \delta_{p_i} \in \mathfrak{M}_x(A)$.
2. Let $T = \mathrm{Th}(\mathbb{N}, =)$, $|x| = 1$. Then

$$S_x(\mathbb{M}) = \{\mathrm{tp}(a/\mathbb{M}) : a \in \mathbb{M}\} \cup \{p_\infty\},$$

   where p_∞ is the unique non-realized type axiomatized by $\{x \neq a : a \in \mathbb{M}\}$. By QE, every formula is a Boolean combination of $\{x = a : a \in \mathbb{M}\}$, from which it follows that every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is as in (1).
3. More generally, if T is ω-stable (e.g. strongly minimal, say ACF_p for p prime or 0) and x is finite, then every $\mu \in \mathfrak{M}_x(\mathbb{M})$ is a sum of types as in (1).
4. Let $T = \mathrm{Th}(\mathbb{R}, <)$, λ be the Lebesgue measure on $\mathbb{R}$ and $|x| = 1$. For $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$, define $\mu(\varphi) := \lambda\left(\varphi(\mathbb{M}) \cap [0,1]_{\mathbb{R}}\right)$ (this set is Borel by QE). Then μ is a Keisler measure, but not a sum of types as in (1).

Independent product of invariant types $\otimes$

- Assume that $p \in S_x(\mathbb{M}), q \in S_y(\mathbb{M})$ and p is $\mathrm{Aut}(\mathbb{M}/\mathcal{M})$-invariant for some small $\mathcal{M} \preceq \mathbb{M}$. Then $p \otimes q \in S_{xy}(\mathbb{M})$ is defined as follows: for every small $\mathcal{M} \preceq \mathcal{N} \prec \mathbb{M}$, we let $p \otimes q|_{\mathcal{N}} = tp(a, b/\mathcal{N})$ for some/any $b \models q|\mathcal{N}$ and $a \models p|_{\mathcal{N}b}$.
- E.g. if p is the non-realized type in $\mathrm{Th}(\mathbb{N}, =)$, then $p(x) \otimes p(y) = p(y) \otimes p(x)$ is axiomatized by

$$\{x \neq a, y \neq a : a \in \mathbb{M}\} \cup \{x \neq y\}.$$

- Assume $p(x) = \{x > a : a \in \mathbb{M}\}$ in $\mathrm{Th}(\mathbb{Q}, <)$. Then

$$p(x) \otimes p(y) = \{x > a, y > a : a \in \mathbb{M}\} \cup \{x > y\} \neq p(y) \otimes p(x).$$

- Hence $\otimes$ is associative, but not commutative (unless T is stable).

Convolution product $*$ of invariant types

- Assume now that T expands a group, i.e. there exists a definable functions $\cdot$ such that for some/any $\mathcal{M} \models T$, $(\mathcal{M}_x, \cdot)$ is a group.
- In this case, given invariant $p, q \in S_x(\mathbb{M})$, we have an invariant type $p * q \in S_x(\mathbb{M})$ via

$$\varphi(x) \in p * q \iff \varphi(x \cdot y) \in p(x) \otimes q(y)$$

  for every $\varphi(x) \in \mathcal{L}_x(\mathbb{M})$.
- Equivalently, $p * q = \operatorname{tp}(a \cdot b/\mathbb{M})$ for some/any $(a, b) \models p \otimes q$ in a larger monster model.
- Given $\mathcal{M} \prec \mathbb{M}$, let $S_x^{\text{inv}}(\mathbb{M}, \mathcal{M})$ be the set of all $\operatorname{Aut}(\mathbb{M}/\mathcal{M})$-invariant global types, and $S_x^{\text{fs}}(\mathbb{M}, \mathcal{M})$ the set of global types finitely satisfiable in $\mathcal{M}$. Then $(S_x^{\dagger}(\mathbb{M}, \mathcal{M}), *)$ is a compact left-continuous semigroup.
- "Left continuous" means: the map $- * q : S_x^{\dagger}(\mathbb{M}, \mathcal{M}) \to S_x^{\dagger}(\mathbb{M}, \mathcal{M})$ is continuous for every fixed $q \in S_x^{\dagger}(\mathbb{M}, \mathcal{M})$.

Idempotent types

- A type $p \in S^{\dagger}_{x}(\mathbb{M}, \mathcal{M})$ is *idempotent* if $p * p = p$.
- E.g. let $\mathcal{M}$ be $(\mathbb{Z}, +, P_{n,\alpha})$, with $(P_{n,\alpha} : \alpha < 2^{\aleph_0})$ naming all subsets of $\mathbb{Z}^n$, for all n.
  Then all types over $\mathcal{M}$ are trivially definable, and idempotent types are precisely the idempotent ultrafilters in the sense of Galvin–Glazer's proof of Hindman's theorem (for every finite partition of $\mathbb{Z}$, some part contains all finite sums of elements of an infinite set), see e.g. [Andrews, Goldbring'18].
- In stable theories, idempotent types are known to arise from type-definable subgroups (group chunk theorem and its variants [Hrushovski, Newelski]).
- This is parallel to the following classical line of research:

Motivation: analogy with the classical (locally-)compact case

- Let G be a locally compact topological group.
- Then the space of regular Borel probability measures on G is equipped with the *convolution product*:

$$\mu * \nu(A) = \int_{y \in G} \int_{x \in G} \chi_A(x \cdot y) d\mu(x) d\nu(y)$$

  for a Borel set $A \subseteq G$.
- If G is compact, then μ is idempotent if and only if the support of μ is a compact subgroup of G and μ restricted to it is the (bi-invariant) Haar measure [Kawada, Itô'40], [Wendel'54].
- Same characterization extends to locally compact abelian groups [Rudin'59, Cohen'60].
- Compact (semi-)topological semigroup — the picture becomes more complicated [Glicksber'59, Pym'69, . . .].

Independent product $\otimes$ of definable Keisler measures

- We would like to find a parallel for Keisler measures, generalizing the situation for types. First, need to make sense of the convolution product.
- A Keisler measure $\mu \in \mathfrak{M}_x(\mathbb{M})$ is *Borel definable* (over $\mathcal{M} \preceq \mathbb{M}$) if:
  1. for any $\varphi(x, y) \in \mathcal{L}_{xy}$ and $b \in \mathbb{M}_y$, $\mu(\varphi(x, b))$ depends only on $\mathrm{tp}(b/\mathcal{M})$
     (in which case, given $q \in S_y(\mathcal{M})$, we write $\mu(\varphi(x, q))$ to denote $\mu(\varphi(x, b))$ for some/any $b \models q$);
  2. the map $q \in S_y(\mathcal{M}) \mapsto \mu(\varphi(x, q)) \in [0, 1]$ is Borel.
- Given $\mu \in \mathfrak{M}_x(\mathbb{M}), \nu \in \mathfrak{M}_y(\mathbb{M})$ with μ Borel definable over $\mathcal{M}$, we can define $\mu \otimes \nu \in \mathfrak{M}_{xy}(\mathbb{M})$ via

$$\mu \otimes \nu(\varphi(x, y)) := \int_{S_y(\mathcal{M})} \mu(\varphi(x, q)) d\nu|_{\mathcal{M}}(q).$$

- The integral makes sense by (2), viewing $\nu|_{\mathcal{M}}$ as a regular Borel measure on $S_y(\mathcal{M})$.

Convolution product $*$ of definable Keisler measures

- We restrict to NIP groups to avoid some technicalities.
- If T is NIP, then every automorphism-invariant measure is Borel-definable, and $\otimes$ on invariant measures extends $\otimes$ on invariant types defined earlier.
- If now T expands a group, given invariant $\mu, \nu \in \mathfrak{M}_x(\mathbb{M})$, we get an invariant $\mu * \nu \in \mathfrak{M}_x(\mathbb{M})$ via

$$\mu * \nu(\varphi(x)) := \mu_x \otimes \nu_y(\varphi(x \cdot y)).$$

- Again, restricting to types, we recover $*$ defined earlier.
- We let $\mathfrak{M}_x^{\mathrm{inv}}(\mathbb{M}, \mathcal{M})$ be the set of global $\mathrm{Aut}(\mathbb{M}/\mathcal{M})$-invariant measures, and $\mathfrak{M}_x^{\mathrm{fs}}(\mathbb{M}, \mathcal{M})$ the set of global measures finitely satisfiable in $\mathcal{M}$.

Theorem (C., Gannon'20)

In an NIP group, $\mathfrak{M}_x^{\dagger}(\mathbb{M}, \mathcal{M})$ is a compact left continuous semigroup.

Idempotent Keisler measures vs the classical locally compact case

- First of all, in general a definable group has no non-discrete topology.
- Given $\mu \in \mathfrak{M}_x(A)$, its *support* is

$$S(\mu) := \{p \in S_x(A) : \varphi(x) \in p \implies \mu(\varphi(x)) > 0\}.$$

  It is a closed non-empty subset of $S_x(A)$.
- As we mentioned, in a locally compact topological group, support of an idempotent measure is a closed subgroup — no longer true for idempotent Keisler measures (with respect to $*$ on types), even if there is some nice topology present.

Supports of idempotent Keisler measures: a theorem

- Adapting Glicksberg, we show:

Theorem (C., Gannon'20)

(T NIP) Let $\mu \in \mathfrak{M}_x(\mathbb{M})$ *be an idempotent definable Keisler measure. Then* $(S(\mu), *)$ *is a compact, left continuous semigroup with no closed two-sided ideals.*

- Where $I \subseteq S(\mu)$ is a left (right) ideal if: $q \in I \implies p * q \in I$ (resp., $q * p \in I$) for every $p \in S(\mu)$. Two-sided = both left and right.

Type-definable subgroups

- Instead of closed subgroups in the topological setting, we consider *type-definable* subgroups.
- Assume that $\mathbb{M} \models T$ expands a group, and $\mathcal{H}$ is a type-definable subgroup of $(\mathbb{M}, \cdot)$ (i.e. the underlying set of $\mathcal{H}$ can be defined by a small partial type $H(x)$ with parameters in $\mathbb{M}$).
- Let $\mathcal{H}$ be type-definable and suppose that $\mu \in \mathfrak{M}_x(\mathbb{M})$ is concentrated on $\mathcal{H}$ (i.e. $p \in S(\mu) \implies p(x) \vdash H(x)$) and is right $\mathcal{H}$-invariant (i.e. for any $\varphi(x) \in \mathcal{L}_x(\mathbb{M}), a \in \mathcal{H}$, $\mu(\varphi(x)) = \mu(\varphi(x \cdot a))$. Then μ is idempotent (and $\mathcal{H}$ is said to be *definably amenable*).
- By analogy with the classical case, we might expect all idempotent Keisler measures in model-theoretically tame groups to be of this form.

Idempotent measures in stable groups

Theorem (C., Gannon'20)

Let T be a stable theory expanding a group and $\mu \in \mathfrak{M}_x(\mathbb{M})$ a Keisler measure. TFAE:

1. *μ is idempotent;*
2. *μ is the unique right/left-invariant measure on its stabilizer, i.e. the type-definable subgroup $St(\mu) = \{g \in \mathbb{M} : g \cdot \mu = \mu\}$.*

- The following groups are stable: abelian, free, algebraic over $\mathbb{C}$ (e.g. $\mathrm{GL}_n(\mathbb{C}), \mathrm{SL}_n(\mathbb{C})$, abelian varieties).
- Ingredients: structure of the supports of definable idempotent measures in NIP; definability measures in stable theories; a variant of Hrushovski's group chunk theorem for partial types due to Newelski.
- Further results: an analog for *generically stable* measures in abelian NIP groups; for G^{00}-invariant measures; in general definably amenable NIP groups — the picture is more complicated.

The structure of the convolution semigroups, 1

Fact (Ellis)

Suppose $(X, *)$ is a left-continuous compact semigroup. Then there exists a minimal (closed) left ideal I. Let $\mathrm{id}(I) = \{u \in I : u^2 = u\}$ be the set of idempotents in I.

1. $\mathrm{id}(I)$ *is non-empty.*
2. *For every $u \in \mathrm{id}(I)$, $u * I$ is a subgroup of I with identity u. Its isomorphism type doesn't depend on I or u, in view of which we refer to $u * I$ as the ideal group.*
3. $I = \bigcup\{u * I : u \in \mathrm{id}(I)\}$, *where the sets in the union are pairwise disjoint.*

The structure of the convolution semigroups, 2

- The so-called "Ellis group conjecture" of Newelski, and Pillay:

Fact (C., Simon)

In a definably amenable NIP group $\mathcal{G}$, with $G \prec \mathcal{G}$, the ideal group of $(S_x^{\mathrm{fs}}(\mathcal{G}, G), *)$ is isomorphic to $\mathcal{G}/\mathcal{G}^{00}$ (where $\mathcal{G}^{00}$ is the smallest type-definable subgroup of bounded index).

- In particular, the ideal group is often non-trivial in this setting.
- The situation is quite different in the convolution semigroups of measures, due to the presence of the convex structure:

The structure of the convolution semigroups, 3

Theorem (C., Gannon)

Assume that $\mathcal{G}$ is NIP, and let I be a minimal left ideal of $\mathfrak{M}_x^\dagger(\mathcal{G}, G)$.

1. *I is a closed convex subset of $\mathfrak{M}_x^\dagger(\mathcal{G}, G)$.*
2. *For any $\mu \in I$, $\pi_*(\mu) = h$, where h is the normalized Haar measure on $\mathcal{G}/\mathcal{G}^{00}$ and $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}^{00}$ is the quotient map.*
3. *If $\mathcal{G}/\mathcal{G}^{00}$ is non-trivial, then I does not contain any types.*
4. *For any idempotent $u \in I$, we have $u * I \cong (e, \cdot)$. In other words, the ideal group is always trivial.*
5. *Every element of I is an idempotent, and $\mu * \nu = \mu$ for all $\mu, \nu \in I$.*

The structure of the convolution semigroups, 4

Theorem (C., Gannon)

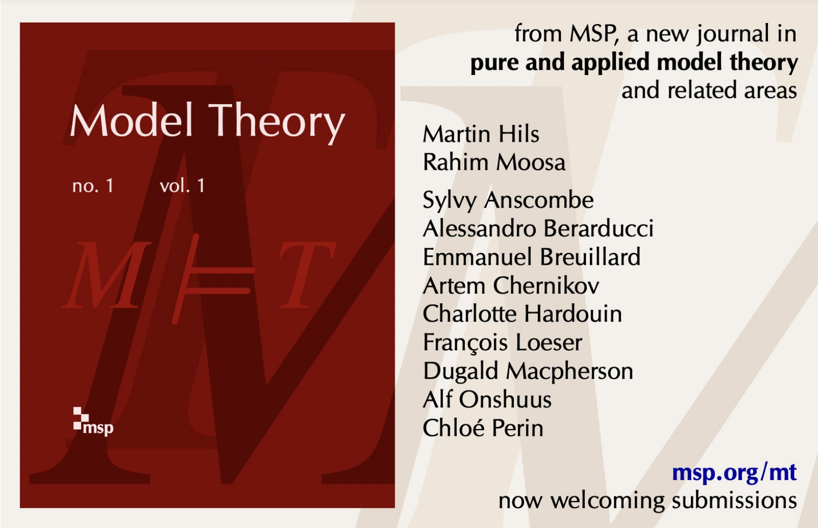
Assume that $\mathcal{G}$ is NIP and definably amenable.

1. *In $\mathfrak{M}^{\mathrm{fs}}_x(\mathcal{G}, G)$, minimal left ideals are of the form $I = \{\nu\}$, where $\nu \in \mathfrak{M}^{\mathrm{fs}}_x(\mathcal{G}, G)$ is a G-left-invariant measure.*
2. *In $\mathfrak{M}^{\mathrm{inv}}_x(\mathcal{G}, G)$, there exists a unique minimal left (and in fact two-sided) ideal*

   $$I = \left\{\mu \in \mathfrak{M}^{\mathrm{inv}}_x(\mathcal{G}, G) : \mu \text{ is } \mathcal{G}\text{-right-invariant}\right\}.$$

   The set $\mathrm{ex}(I)$ of extreme points of I is closed (hence I is a Bauer simplex) and equal to $\left\{\mu_p : p \in S^{\mathrm{inv}}_x(\mathcal{G}, G) \textit{ is right } f\textit{-generic}\right\}$.
3. *If $\mathcal{G}$ is fsg and $\mu \in \mathfrak{M}_x(\mathcal{G})$ is the unique $\mathcal{G}$-left-invariant measure, then $I = \{\mu\}$ is the unique minimal left (in fact, two-sided) ideal in both $\mathfrak{M}^{\mathrm{inv}}_x(\mathcal{G}, G)$ and $\mathfrak{M}^{\mathrm{fs}}_x(\mathcal{G}, G)$.*
4. *If $\mathcal{G}$ is not definably amenable, then $\mathrm{ex}(I)$ is infinite.*

Thank you!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