Fractional Helly property in model theory

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Helly's theorem

Theorem

[Helly, 1913] Let $S_1, \ldots, S_n$ be convex subsets of $\mathbb{R}^d$, with $n > d$. If the intersection of every $d+1$ of these sets is non-empty, then the intersection of the whole collection $\bigcap_{i=1}^{n} S_i$ is non-empty.

Fractional Helly's theorem

Theorem

[Katchalski, Liu, 1979] Fix dimension $d \geq 1$*. The for every* $\alpha \in (0, 1]$ *there exists* $\beta = \beta(\alpha, d) \in (0, 1]$ *such that the following holds:*
If $S_1, \ldots, S_n$ *are convex sets in* $\mathbb{R}^d$*,* $n \geq d + 1$*, such that* $\bigcap_{i \in I} S_i \neq \emptyset$ *for at least* $\alpha \binom{n}{d+1}$ *of the sets* $I \in \binom{[n]}{d+1}$*, then there is some* $J \subseteq [n]$ *such that* $|J| \geq \beta n$ *and* $\bigcap_{i \in J} S_i \neq \emptyset$*.*

FHP for set systems

- Let now $(X, \mathcal{F})$ be an arbitrary set system (i.e. X is a set and $\mathcal{F}$ is a family of subsets of X).

Definition

We say that $\mathcal{F}$ satisfies the *fractional Helly property*, or FHP, if there is some $d \in \mathbb{N}$ such that for every $\alpha \in (0,1]$ there exists $\beta \in (0,1]$ satisfying the following:
If $(S_1, \ldots, S_n) \in \mathcal{F}^n$ is such that $\bigcap_{i \in I} S_i \neq \emptyset$ for at least $\alpha\binom{n}{d}$ of the sets $I \in \binom{[n]}{d}$, then there is some $J \subseteq [n]$ such that $|J| \geq \beta n$ and $\bigcap_{i \in J} S_i \neq \emptyset$.
The minimal d for which this holds is the *fractional Helly number* for $\mathcal{F}$ (if this holds for d, then also holds for any $d' \geq d$).

FHP for formulas

- Let T be a complete first-order theory in a language $\mathcal{L}$, $\mathcal{M} \models T$ and $\phi(x, y) \in \mathcal{L}$ a formula.
- We associate with it a definable family $\mathcal{F}_\phi = \{\phi(M, b) : b \in M_y\}$ of subsets of M_x.

Definition

We say that $\phi(x, y)$ has FHP if the family $\mathcal{F}_\phi$ has FHP (and the fractional Helly number of ϕ is the fractional Helly number of $\mathcal{F}_\phi$).
T has FHP if every formula has FHP.
(Note: FHP is a property of the theory, rather than of the specific model M).

FHP and Shelah's classification

- FHP implies NTP_2.
- FHP implies low.
- [Matousek, 2003] NIP implies FHP. More precisely, if $\pi^*_{\mathcal{F}}(n) = o\left(n^d\right)$ as $n \to \infty$ (e.g. if $vc^*(\mathcal{F}) < d$), then d is a fractional Helly number for $\mathcal{F}$.
- If all formulas $\phi(x, y)$ with $|x| = 1$ have FHP, then T has FHP.
- There is no implication between FHP and wnfcp.

FHP relatively to a class of measures

- Recall: a Keisler measure μ on M_y is a finitely additive probability measure on the Boolean algebra of definable subsets of M_y.
- Let $\mathfrak{M}$ be a class of measures on M_y such that for every $n \in \mathbb{N}$ and $\mu_1, \ldots, \mu_n \in \mathfrak{M}$ we fix a certain product measure μ^* on $M_{n \times y}$. If $\mu_i = \mu$ for $i = 1, \ldots, n$ we denote μ^* by $\mu^{(n)}$.

Definition

We say that $\phi(x, y)$ has *FHP relatively to* $\mathfrak{M}$ if there is some $d \in \mathbb{N}$ such that for any $\alpha > 0$ there is $\beta > 0$ satisfying:
for any $\mu \in \mathfrak{M}$, if $\mu^{(d)}\left(\exists x \bigwedge_{i=1}^{d} \phi(x, y_i)\right) \geq \alpha$ then there is some $a \in M_x$ with $\mu(\phi(a, y)) \geq \beta$.

- Note: ϕ has FHP iff it has FHP relatively to the class $\mathfrak{M}_{\text{fin}}$ of finitely supported measures (with the unique product measure).

Fap measures, 1

Definition

Let μ be a measure on M_x and $\phi(x, y) \in \mathcal{L}$.

1. Given $\varepsilon > 0$, a multiset $A = \{a_1, \ldots, a_n\} \subseteq M_x$ is an *ε-approximation* of μ *on* ϕ if for every $b \in M_y$, $\mu(\phi(x, b)) \approx^{\varepsilon} \frac{|\{i : \models \phi(a_i, b)\}|}{n}$.
2. μ is *finitely approximated on* ϕ, or *fap on* ϕ, if it admits a finite ε-approximation on ϕ for all $\varepsilon > 0$.
3. μ is *fap* if it is fap on every ϕ.

- In NIP, μ is fap iff μ is generically stable.
- Examples of fap measures:
  - A measure concentrated on a finite set,
  - In an *o*-minimal M, Lebesgue measure on $[0, 1]$ (restricted to definable sets),
  - In $\mathbb{Q}_p$, additive Haar measure on the compact ball $\mathbb{Z}_p$.
- The $\{0, 1\}$-measure given by the type at $+\infty$ in $(\mathbb{R}, +, \times, <, 0, 1)$ is not fap.

Fap measures, 2

- Assume we are given a definable relation $E(x, y) \in \mathrm{Def}(M_{xy})$.
- Let μ and ν be Keisler measures on M_x and M_y, respectively.
- Note that $\mathrm{Def}(M_{xy}) \neq \mathrm{Def}(M_x) \times \mathrm{Def}(M_y)$, and E may not be $\mu \times \nu$-measurable.
- In general, there are many ways to extend the product measure $\mu \times \nu$ to a measure ω on $\mathrm{Def}(M_{xy})$.
- For fap measures, we have a canonical choice.

Definition. Given fap measures μ, ν, on M_x, M_y respectively, we define a measure $\mu \otimes \nu$ on M_{xy} by

$$\mu \otimes \nu (E(x, y)) = \int_{M_x} \left(\int_{M_y} \mathbf{1}_E(x, y)\, d\nu \right) d\mu.$$

- It is well-defined, fap and satisfies the Fubini property: $\mu \otimes \nu = \nu \otimes \mu$.

FHP relatively to fap measures

Lemma

If ϕ has FHP (i.e., relatively to the class $\mathfrak{M}_{fin}$), then ϕ has FHP relatively to the class $\mathfrak{M}_{fap}$ of fap measures (with $\otimes$).

- Then Matousek's theorem + Fact immediately imply (taking contrapositives):

Fact

[Hrushovski, Pillay, Simon, "A note on generically stable measures and fsg groups"] Let T be NIP *and μ a generically stable measure. If $\mu(\phi(x, b)) = 0$ for all b, then there is some d such that $\mu^{(d)}(\exists y(\phi(x_1, y) \wedge \ldots \wedge \phi(x_d, y))) = 0$.*

- Conversely, under the global NIP assumption Matousek's theorem follows from the fact using that the class of generically stable measures in NIP is closed under ultraproducts.

Colorful version

- No reason to fix only one measure. We have (refining [Pillay, "Weight and measure in NIP theories"]):

Theorem

Let T be NIP, such that dp-rank("$x = x$") $\leq d$. Then for any formulas $\phi_1(x, y_1), \ldots, \phi_{d+1}(x, y_{d+1}) \in \mathcal{L}$ and $\alpha > 0$ there is some $\beta > 0$ such that:
if μ_i is a fap measure on M_{y_i}, $i = 1, \ldots, d+1$, and $\mu_1 \otimes \ldots \otimes \mu_{d+1}\left(\exists x \bigwedge_{i=1}^{d+1} \phi_i(x, y_i)\right) \geq \alpha$ then there is some $a \in M_x$ and $1 \leq i \leq d+1$ such that $\mu_i(\phi_i(a, y_i)) \geq \beta$.

- This corresponds to the so called "colorful fractional Helly property" for NIP families (was known in combinatorics for convex sets in $\mathbb{R}^k$, due to [Barany et. al., 2014]).

Corollary

The fractional Helly number of $\phi(x, y)$ is at most dp-rank($x = x$) $+ 1$ (by the Theorem for $\phi_i(x, y_i) = \phi(x, y)$ and $\mu_i = \mu$).

(p, q)-theorem

Definition

We say that a family of sets $\mathcal{F} \subseteq \mathcal{P}(X)$ is *pierceable* if there is some number $d \in \omega$ such that for any $q \geq p \geq d$ there is some $N = N(p, q) \in \omega$ such that if a finite subfamily $\mathcal{F}' \subseteq \mathcal{F}$ satisfies the (p, q)-*property* (i.e. among any q sets from $\mathcal{F}$, at least p have a non-empty intersection), then there are some $a_1, \ldots, a_N \in X$ such that every $S \in \mathcal{F}'$ contains at least one of the a_i's.

Theorem

[Alon, Kleitman] For any d, $\mathcal{F} = \left\{\text{convex subsets of } \mathbb{R}^d\right\}$ is pierceable.

Theorem

[Matousek] If $VC(\mathcal{F}) < \infty$, then $\mathcal{F}$ is pierceable.

- The proof combines FHP + existence of ε-nets.

(p, q)-theorem and NIP

Theorem

Assume that $VC(\mathcal{F}) = \infty$. Then the family $\mathcal{F}' = \{S_1 \wedge \neg S_2 : S_1, S_2 \in \mathcal{F}\}$ is not pierceable.

- In particular, T is NIP iff $\mathcal{F}_\phi$ is pierceable for every $\phi \in \mathcal{L}$.
- The family of convex sets in $\mathbb{R}^d$ shows that this doesn't hold at the level of a formula.

FHP in MS-measurable structures, 1

Definition

[Macpherson, Steinhorn, 2008] An L-structure M is *MS-measurable* if for every non-empty set $X \subseteq M^n$ definable with parameters, we have a pair $(\dim(X), \text{meas}(X))$ with $\dim(X) \in \mathbb{N}$, $\dim(X) \leq n$ and $\text{meas}(X) \in \mathbb{R}_{>0}$ satisfying some strong definability properties and a **Fubini** condition.

Example

[Chatzidakis, van den Dries, Macintyre] Let $M = \prod_{p \in P} \mathbb{F}_p / \mathcal{U}$ be an ultraproduct of finite fields, P a set of primes, $\mathcal{U}$ a non-principal ultrafilter on P.

Let $X \subseteq M^n$ be a definable set, $X = \prod_{p \in P} X_p / \mathcal{U}$ for some $X_p \subseteq \mathbb{F}_p^n$. Then $(\dim(X), \text{meas}(X)) = (d, \alpha)$ if

$$|X_p| \approx \alpha p^d$$

for $\mathcal{U}$-many p.

FHP in MS-measurable structures, 2

- For any definable set $B \subseteq M_y$, we have a Keisler measure μ_B concentrated on B and defined by

$$\mu_B(X) = \begin{cases} \frac{\text{meas}(X \cap B)}{\text{meas}(B)} & \text{if } \dim(X \cap B) = \dim(B), \\ 0 & \text{if } \dim(X \cap B) < \dim(B) \end{cases}$$

  for all definable $X \subseteq M_y$.
- Let $\mathfrak{M} = \{\mu_B : B \subseteq M_y \text{ definable}\}$, and we take $\mu_{B_1} \otimes \mu_{B_2} := \mu_{B_1 \times B_2}$ (given by the same double integral as in fap measures).

Theorem

In an MS-measurable theory, $\phi(x, y)$ satisfies FHP relatively to the class of measures $\mathfrak{M}_y = \{\mu_B : B \subseteq M_y \text{ definable}\}$.

- In particular, MS-measurable implies FHP (as μ_B with B finite is the counting measure concentrated on B), and the fractional Helly number of $\phi(x, y)$ is at most $\max\{\dim \phi(x, b) : b \in M_y\} + 1$.
- Gives a definable FHP theorem for large finite fields.

Ultraproducts of the p-adics

- For each prime p, the field $\mathbb{Q}_p$ is NIP, so satisfies FHP relatively to generically stable measures.

Theorem

Let $\mathcal{M} = \prod_{p \in P} \mathbb{Q}_p / \mathcal{U}$ for P a set of primes, $\mathcal{U}$ a non-principal ultrafilter on P. Then $T = \mathrm{Th}(\mathcal{M})$ satisfies FHP (i.e. for finitely supported measures).

- Problem: is there a motivic version?
- Note: T is neither NIP nor simple. Previously known: T is NTP_2 (C.) and moreover T is inp-minimal (C., Simon).

Some more examples

Theorem

1. Th $(\mathbb{Z}, +, Sqf)$ *is FHP (elaborating on the results of Bhardwaj-Tran).*
2. *Assuming Dickson's conjecture,* Th $(\mathbb{Z}, +, Pr)$ *is not FHP (but it is supersimple of SU-rank* 1 *by Kaplan-Shelah, hence wnfcp).*

f-generics in amenable FHP groups

- Let G be a group and $A \subseteq G$.
- A is generic $\implies$ A is weakly generic $\overset{G \text{ amenable}}{\implies}$ $\mu(A) > 0$ for some G-invariant measure μ $\implies$ A is f-generic.
- [C., Simon] In defianbly amenable NIP, f-generic = weak generic.

Theorem

Let G be an amenable group with FHP and $A \subseteq G$ definable. TFAE:

1. *A is f-generic.*
2. *$\mu(A) > 0$ for some G-invariant Keisler measure.*
3. *$\mu(A) > 0$ for some G-invariant measure on $\mathcal{P}(G)$.*

- Problem: does (1) $\iff$ (2) hold assuming only that G is definably amenable?

f-generics in amenable FHP groups

Example

1. In $\mathrm{Th}(\mathbb{Z}, +, \mathrm{Pr})$, Pr is f-generic, but $\mu(\mathrm{Pr}) = 0$ for any invariant Keisler measure (by the Prime Number theorem).
2. In $\mathrm{Th}(\mathbb{Z}, +, \mathrm{Sqf})$, $\mu(\mathrm{Sqf}) = \frac{6}{\pi^2} > 0$ for an invariant measure (Banach density), but Sqf is not weakly generic.